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On Some Problems of the Spectral Theory

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Contents

Introduction	3
1 Asymptotics, Basicity and Expansion	14
1.1 Expansion Theorems	14
1.2 Asymptotics of the Eigenvalues	21
1.3 Asymptotics of the Norming Constants	26
1.4 Riesz Bases Generated by the Spectra of Sturm-Liouville Problems	38
2 Constructive Solution of an Inverse Sturm-Liouville Problem	46
2.1 Derivation of an Analogue of the Gelfand-Levitan Equation . . .	46
2.2 The Constructive Solution of the Inverse Problem	52
2.3 Implementation of the Algorithm	61
2.4 Appendix	61
3 Zeros of Eigenfunctions and Sturm Oscillation Theorem	66
3.1 On the “movement” of the zeros of eigenfunctions	66
Conclusion	73
Acknowledgements	74
Bibliography and Author’s Publications	75

Introduction

“La résolution de la plupart des problèmes relatifs à la distribution de la chaleur dans des corps de formes diverses et aux petits mouvements oscillatoires des corps solides élastiques, des corps flexibles, des liquides et des fluides élastiques, conduit à des équations différentielles linéaires du second ordre...”

This is an excerpt from Charles Sturm’s motivation presented in the first large paper in 1836 (see [1, p. 392]) devoted to a new subject in mathematical analysis, later known as Sturm-Liouville theory. From translation into English it sounds like this:

“The solvability of the most of problems related to the distribution of heat in the bodies of various shapes and small oscillatory movements of solid elastic, flexible body, liquids and elastic fluids, leads to the linear second order differential equations...”

In his second paper in 1836 (see [2]), it was explained in more detail how the partial differential equations arising from the problems above can be solved by separating the variables, leading in general to a second-order ordinary differential equation with a parameter. The parameter must be chosen so that certain boundary conditions are satisfied.

It is worthwhile to mention some events that preceded the appearance of these papers and the Sturm-Liouville theory itself:

- 1) Eigenvalue problems for linear second-order differential equations with imposed boundary conditions had turned up in early 18th century in the study of vibratory motions (see papers of Brook Taylor [3] on the vibrating strings, Johann Bernoulli [4] on the hanging chain and Daniel Bernoulli [5, 6] on the vibrating hanging homogeneous chain).
- 2) From 1747 onwards d’Alembert and Euler had derived the partial differential equations describing vibrating strings, chains and membranes (see [7–9]) and obtained the eigenvalue problem by separating variables.
- 3) In 1821 in his lectures at the École Polytechnique Cauchy had proved the existence of a

solution of a first order differential equation with given initial conditions (see [10]).

- 4) In 1822 the technique of separating variables had been introduced by Fourier (see [11]) (from 1807 was widely used in the theory of heat, first by Fourier, and soon thereafter by almost all the younger French mathematicians).

This vast complex of research presented ample motivation for Charles Sturm and Joseph Liouville, who create a theory that deals with the general linear second-order differential equation

$$-\frac{d}{dt} \left(p(t) \frac{du}{dt} \right) + l(t) u(t) = \nu r(t) u(t), \quad a \leq t \leq b, \quad (0.1)$$

with the imposed boundary conditions

$$u(a) \cos \alpha + u'(a) \sin \alpha = 0, \quad \alpha \in (0, \pi], \quad (0.2)$$

$$u(b) \cos \beta + u'(b) \sin \beta = 0, \quad \beta \in [0, \pi), \quad (0.3)$$

where $p(t) > 0$, $r(t) > 0$ and $l(t)$ are given functions on $[a, b]$, ν is a parameter.

The questions studied by Sturm and Liouville within the new theory can be roughly divided into three groups:

- 1) properties of the eigenvalues,
- 2) qualitative behavior of the eigenfunctions,
- 3) expansion of arbitrary functions in an infinite series of eigenfunctions.

Sturm suggested and investigated 1) and 2), Liouville studied 3) with further development for 1) and 2) (see [1, 2, 12–17]). For the detailed discussion of these results we refer the reader to the two papers of Lützen ([18, 19]), where the place and significance of Charles Sturm and Joseph Liouville contribution in the history of mathematics in the 19th century are considered in details.

The second stage of the development of the Sturm-Liouville theory happened at the end of the 19th century and the beginning of the 20th century and connected with the appearance of two theories. First one is the spectral theory for various classes of differential and integral operators, which arose from the works of Poincaré, Fredholm, Lebesgue, Hilbert, Fréchet,

Schmidt, F. Riesz, Weyl, von Neumann and other mathematicians (see [20, 21]). New methods allowed V.A. Steklov to get the rigorous proof that a twice differentiable function satisfying the boundary conditions (0.2) and (0.3) can be expanded in as an absolutely and uniformly convergent generalized Fourier series for eigenfunctions of the problem (0.1)–(0.3) (see [22, 23]). In 1910 Weyl, relying on the theory of integral equations had developed by Hilbert, constructed the spectral theory of Sturm-Liouville operators given on an infinite interval (see [24]).

The second theory is the quantum mechanics, which arose from the works of Planck, Einstein, Schrödinger, Heisenberg, Born and others. It turned out that Schrödinger equation (see [25]) (one of the basic equations in the quantum mechanics) is closely related the Sturm-Liouville equation (see [26, 27]). Particularly, the one-dimensional time-independent Schrödinger equation is a special case of a Sturm-Liouville equation. In 1929 the inverse Schrödinger problem stimulated Armenian astrophysicist V.A. Ambarzumian to study a uniqueness question for eigenvalues of Sturm-Liouville equation ([28]). This was the first step in the inverse spectral theory for Sturm-Liouville problems.

Another important step in the development of the Sturm-Liouville theory was made in 1967, when Gardner, Greene, Kruskal and Miura (see [29]) related the solution of the Cauchy initial value problem for the Korteweg-de Vries equation to the inverse scattering problem for a one-dimensional linear Schrödinger equation. It turned out that inverse Sturm-Liouville theory is an important tool for a solution of large class non-linear evolution equations (NLEE) ([30, 31]).

Despite the fact, that more than 180 years different generations of mathematicians have contributed to the development of the Sturm-Liouville theory it has not lost its relevance. The recent research, especially in the theory of inverse Sturm-Liouville problems, shows that, in addition to such classical problems as vibrations of strings, Schrödinger equation in quantum mechanics and Korteweg-de Vries equation in the theory of nonlinear waves, Sturm-Liouville problems arise in the theory of plasma dynamics (see [32]), in the biomedical engineering (see [33]) and in many other branches of applied sciences.

In addition to the above, it is noteworthy that in 2005, W.N. Everitt presented a catalogue,

which includes more than 50 examples of Sturm-Liouville differential equations (see [34]), many of which are connected with well-known special functions, and with problems in mathematical physics and applied mathematics.

Our research is devoted to the study of some direct and inverse spectral problems for regular Sturm-Liouville operators. Let us denote by $L(q, \alpha, \beta)$ the following Sturm-Liouville boundary value problem

$$-y'' + q(x)y = \mu y \equiv \lambda^2 y, \quad x \in (0, \pi), \quad \mu \in \mathbb{C}, \quad (0.4)$$

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0, \quad \alpha \in (0, \pi], \quad (0.5)$$

$$y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \quad \beta \in [0, \pi), \quad (0.6)$$

where the potential q is a real-valued, summable function on $[0, \pi]$ (we write $q \in L^1_{\mathbb{R}}[0, \pi]$).

Let us note that the problem (0.4)–(0.6) can be reduced from more general problem (0.1)–(0.3), by applying corresponding Liouville substitution, provided that $p(x)$ and $(p(x)r(x))'$ are absolutely continuous functions (see [35]).

By $L(q, \alpha, \beta)$ we also denote the self-adjoint operator generated by the problem (0.4)–(0.6) in Hilbert space $L^2[0, \pi]$ (see [35, 36]). It is well-known, that the spectrum of $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues (see [35–37]), which we denote by $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, emphasizing the dependence on q , α and β . We assume that eigenvalues μ_n are enumerated in increasing order:

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \dots < \mu_n(q, \alpha, \beta) < \dots .$$

Let $\varphi(x, \mu, \alpha)$ and $\psi(x, \mu, \beta)$ be the solutions of (0.4), satisfying the initial conditions

$$\varphi(0, \mu, \alpha) = \sin \alpha, \quad \varphi'(0, \mu, \alpha) = -\cos \alpha, \quad (0.7)$$

$$\psi(\pi, \mu, \beta) = \sin \beta, \quad \psi'(\pi, \mu, \beta) = -\cos \beta. \quad (0.8)$$

It is well-known ([35, 36, 38]) that for fixed x , the functions $\varphi, \varphi', \psi, \psi'$ are entire with respect to μ . We set by $W_{\alpha, \beta}(x, \mu)$ Wronskian of the solutions $\varphi(x, \mu, \alpha)$ and $\psi(x, \mu, \beta)$:

$$W_{\alpha, \beta}(x, \mu) := \varphi(x, \mu, \alpha) \psi'(x, \mu, \beta) - \varphi'(x, \mu, \alpha) \psi(x, \mu, \beta). \quad (0.9)$$

By virtue of Liouville formula for the Wronskian (see, e.g., [39]) $W_{\alpha,\beta}(x, \mu)$ does not depend on x , i.e.

$$W_{\alpha,\beta}(\mu) = W_{\alpha,\beta}(x, \mu) = W_{\alpha,\beta}(\pi, \mu) = W_{\alpha,\beta}(0, \mu). \quad (0.10)$$

The eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, of $L(q, \alpha, \beta)$ are the solutions of the equation $W_{\alpha,\beta}(\mu) = 0$ (see, e.g., [37]). It is easy to see that the functions $\varphi_n(x) := \varphi(x, \mu_n, \alpha)$ and $\psi_n(x) := \psi(x, \mu_n, \beta)$, $n = 0, 1, 2, \dots$, are the eigenfunctions corresponding to the eigenvalue μ_n . Since the eigenvalues are simple, then there exist the numbers $\beta_n = \beta_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, such that

$$\psi_n(x) = \beta_n \varphi_n(x), \quad \beta_n \neq 0. \quad (0.11)$$

The squares of the L^2 norms of these eigenfunctions:

$$a_n = a_n(q, \alpha, \beta) = \int_0^\pi \varphi_n^2(x) dx, \quad b_n = b_n(q, \alpha, \beta) = \int_0^\pi \psi_n^2(x) dx, \quad (0.12)$$

are called norming constants.

The existence, countability and asymptotic formulae for the eigenvalues, asymptotic formulae for the norming constants, completeness of the eigenfunctions in $L^2[0, \pi]$, convergence of expansion for eigenfunctions for various classes of functions, which are the ingredients of the direct Sturm-Liouville problem, have been studied since the mid-19th and early 20th century (see [1, 2, 12–17, 22, 23, 40–43]) and have received a fairly complete form at the end of 20th century thanks to the works of Titchmarsh, Levitan, Marchenko and others in those case when $q \in L^2_{\mathbb{R}}[0, \pi]$, $\sin \alpha \neq 0$, $\sin \beta \neq 0$ and $\alpha = \pi$, $\beta = 0$ (see [35, 37, 44–49]).

Inverse Sturm-Liouville problems, as it was mentioned above, originates from one work of Ambarzumian (dated 1929), where the following uniqueness theorem was proved:

Theorem 0.1 ([28]) *If the eigenvalues $\mu_n \left(q, \frac{\pi}{2}, \frac{\pi}{2} \right) = n^2$, $n \geq 0$, then $q(x) = 0$ a.e. on $(0, \pi)$.*

The first, who pointed out the importance of the result of Ambarzumian, was Swedish mathematician Göran Borg. He showed [50] that Ambarzumian's result is an exception from the "rule" - one spectrum does not determine a Sturm-Liouville operator. In the same paper, Borg

showed that two spectra of a Sturm-Liouville operator (for different boundary conditions) determine it uniquely.

In 1950 Marchenko, using transformation operators (for arbitrary Sturm-Liouville equations constructed by Povzner [51]), proved that the so-called spectral function determines the operator uniquely (see [48, 52]). In the case of regular operator (defined on a finite interval for summable potential) the spectral function is defined by two sequences - eigenvalues and the norming constants.

Other types of uniqueness theorems were studied by Levinson ([53]), Hochstadt and Lieberman ([54]), Isaacson and Trubowitz ([55]), McLaughlin and Rundell ([56]), McLaughlin ([57]), Harutyunyan ([58]) and others.

In 1951 in their fundamental work [59] Gelfand and Levitan succeeded in giving an efficient algorithm for constructive solution of inverse Sturm-Liouville problem through the spectral function for twice differentiable potential. In the case of regular operator their result provides necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$, to be the spectrum and the norming constants of a problem $L(q, \alpha, \beta)$, $\alpha, \beta \in (0, \pi)$, respectively. Afterwards this result was generalized (for more general classes of potentials and for different types of given spectral data) by descendants of Gelfand and Levitan and by other mathematicians (see [45, 60–62] and the references therein). Here we present two of them that are important in the context of our work:

Theorem 0.2 ([60]) *For real numbers $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectral data (spectrum and norming constants) for a certain boundary value problem $L(q, \alpha, \beta)$, with $q(x) = \frac{dF(x)}{dx}$, where F is a bounded variation function on $[0, \pi]$ (we write $F \in BV[0, \pi]$), $\lim_{x \rightarrow 0^+} F(x) = F(0)$ and $\lim_{x \rightarrow \pi^-} F(x) = F(\pi)$, and $\alpha, \beta \in (0, \pi)$, it is necessary and sufficient that the relations*

$$\sqrt{\mu_n} \equiv \lambda_n = n + \frac{\omega}{\pi n} + l_n, \quad \mu_n \neq \mu_m \quad (n \neq m), \quad \sum_{n=1}^{\infty} l_n \sin nx = l(x) \in BV[0, \pi], \quad (0.13)$$

$$\tilde{a}_n \equiv \frac{a_n}{\sin^2 \alpha} = \frac{\pi}{2} + \alpha_n, \quad a_n > 0, \quad \sum \alpha_n \cos nx = \alpha(x) \in BV[0, \pi] \quad (0.14)$$

where ω is a constant, hold.

Theorem 0.3 ([62]) For real numbers $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ to be the spectral data (spectrum and norming constants) for a certain boundary value problem $L(q, \alpha, \beta)$, with $q \in L^2_{\mathbb{R}}[0, \pi]$ and $\alpha, \beta \in (0, \pi)$, it is necessary and sufficient that the relations

$$\sqrt{\mu_n} \equiv \lambda_n = n + \frac{\omega}{\pi n} + \frac{k_n}{n}, \quad \mu_n \neq \mu_m \quad (n \neq m), \quad \{k_n\} \in l_2 \quad (0.15)$$

$$\tilde{a}_n \equiv \frac{a_n}{\sin^2 \alpha} = \frac{\pi}{2} + \frac{k_{n1}}{n}, \quad a_n > 0, \quad \{k_{n1}\} \in l_2, \quad (0.16)$$

where ω is a constant, hold.

Another interesting approach for studying inverse Sturm-Liouville problems was suggested by E. Trubowitz and his colleagues (see [55, 61, 63, 64]). This approach is connected with the description of all problems of the form (0.4)–(0.6) that have the same spectrum (also called isospectrality problem). One of the main ingredients here is the classical Darboux transformation [30, 65], which allows authors in [55, 63] to reduce the problem of the form (0.4)–(0.6) ($\sin \alpha \neq 0, \sin \beta \neq 0$) to the equation of the form (0.4) with the Dirichlet boundary conditions ($y(0) = 0, y(\pi) = 0$) (Dirichlet problem). The whole book [61] is devoted to the study of the Dirichlet problem. Some aspects of this approach were also studied for the case $\sin \alpha = 0, \sin \beta \neq 0$ ($\alpha = \pi, \beta \in (0, \pi)$) (see [64, 66, 67]).

In 1997, Jodeit and Levitan (see [68]) suggested to use Gelfand-Levitan equation (see [59]) and transformation operators (see [51]) to deal with isospectrality problem. They themselves solved the isospectrality problem (explicit formulas are given for parameters determining boundary conditions, implicit formula is given for potential) under assumptions $q' \in L^2_{\mathbb{R}}[0, \pi]$ and $\sin \alpha \neq 0, \sin \beta \neq 0$ ($\alpha, \beta \in (0, \pi)$), with some remarks for the other cases at the end of the paper [68].

Despite the extensive study in the field of the inverse Sturm-Liouville problems, to our knowledge, so far, the necessary and sufficient conditions for the sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ to be the spectrum and the norming constants for the problem $L(q, \pi, \beta)$ (analogously for $L(q, \alpha, 0)$) with $q \in L^1_{\mathbb{R}}[0, \pi]$ have not been found, which in particular means that the constructive solution of the inverse Sturm-Liouville problem in this case is not given. Our primary goal within this work is to find these conditions and provide an efficient algorithm for the constructive solution of the inverse problem.

Before formulation of the main result of the work, we recall some concepts and results that will be used further.

In the paper [69] Harutyunyan, while studying the dependence of the eigenvalues on parameters α and β determining the boundary conditions (0.5) and (0.6), introduced the concept of the function of $\delta_n(\alpha, \beta)$, which is defined by the formula

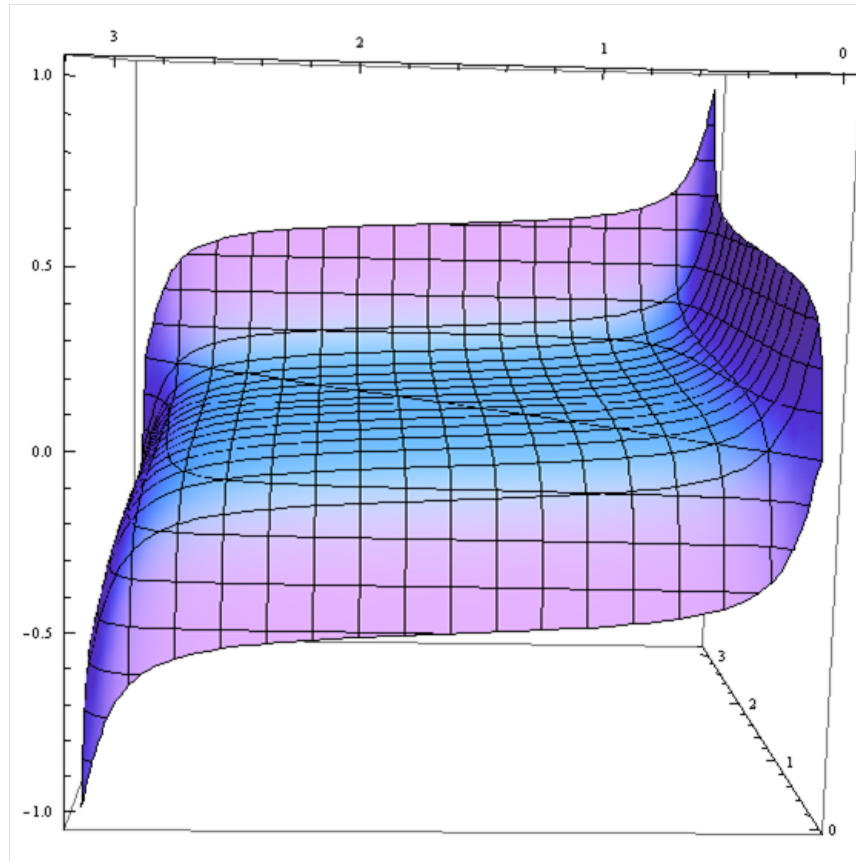
$$\delta_n(\alpha, \beta) := \sqrt{\mu_n(0, \alpha, \beta)} - n = \lambda_n(0, \alpha, \beta) - \lambda_n\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \quad n \geq 2, \quad (0.17)$$

and proved that $-1 \leq \delta_n(\alpha, \beta) \leq 1$ and it is a solution of the following transcendental equation:

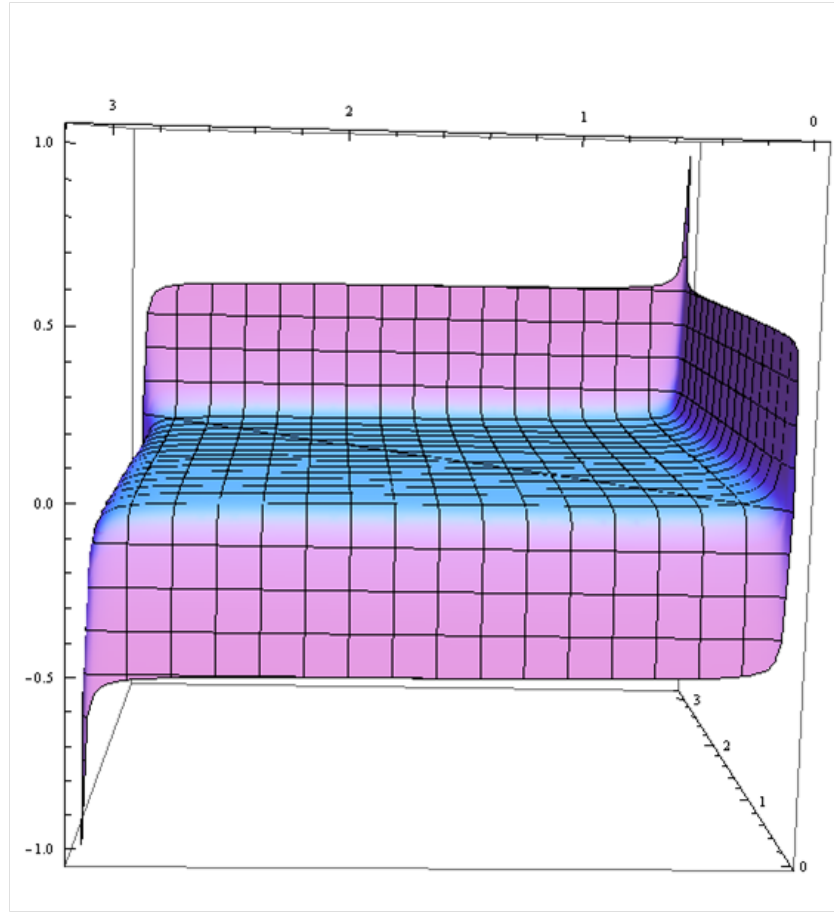
$$\delta = \frac{1}{\pi} \arccos \frac{\cos \alpha}{\sqrt{(n + \delta)^2 \sin^2 \alpha + \cos^2 \alpha}} - \frac{1}{\pi} \arccos \frac{\cos \beta}{\sqrt{(n + \delta)^2 \sin^2 \beta + \cos^2 \beta}}. \quad (0.18)$$

At the end of the work, in the Section 2.4 we will prove that the transcendental equation (0.18) has a unique solution for each fixed $n = n_0 \geq 2$, $\alpha = \alpha_0 \in (0, \pi]$, $\beta = \beta_0 \in [0, \pi)$.

But for now, we provide two plots of $\delta_n(\alpha, \beta)$ function (for $n = 10$ and $n = 100$), which were obtained using Wolfram Mathematica.



Plot 1 ($\delta_{10}(\alpha, \beta)$)



Plot 2 ($\delta_{100}(\alpha, \beta)$)

As it can be seen in the plots the case $\alpha = \pi, \beta \in (0, \pi)$ essentially differs from the well-studied case $\alpha, \beta \in (0, \pi)$. Particularly, $\delta_n(\pi, \beta) \rightarrow \frac{1}{2}$ for $\beta \in (0, \pi)$ (when $n \rightarrow \infty$), which implies that the principal term of asymptotics of the eigenvalues $\mu_n(q, \pi, \beta)$ is $\left(n + \frac{1}{2}\right)^2$, but not n^2 as in (0.13) and (0.15), and the principal term of asymptotics of the norming constants $a_n(q, \pi, \beta)$ is $\frac{\pi}{2\left(n + \frac{1}{2}\right)^2}$, but not $\frac{\pi}{2}$ as in (0.14) and (0.16) (for the detailed explanation see [69]).

Now, let us formulate the main result of the work.

Theorem 0.4 *For two real sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectrum and the norming constants of a problem $L(q, \pi, \beta)$, with $q \in L_{\mathbb{R}}^1[0, \pi]$ and some $\beta \in (0, \pi)$, it is necessary and sufficient that the following relations hold:*

$$\sqrt{\mu_n} \equiv \lambda_n = n + \delta_n(\pi, \beta) + \frac{c}{2(n + \delta_n(\pi, \beta))} + l_n, \quad \mu_n \neq \mu_m \quad (n \neq m), \quad (0.19)$$

$$a_n = \frac{\pi}{2(n + \delta_n(\pi, \beta))^2} \left(1 + \frac{2s_n}{\pi(n + \delta_n(\pi, \beta))} \right), \quad a_n > 0, \quad (0.20)$$

where c is a constant, the remainders $l_n = o\left(\frac{1}{n}\right)$ (when $n \rightarrow \infty$) are such that, the function

$$l(t) = \sum_{n=2}^{\infty} l_n \sin(n + \delta_n(\pi, \beta)) t \quad (0.21)$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$ (we will write $l \in AC(0, 2\pi)$) and the remainders $s_n = o(1)$ (when $n \rightarrow \infty$) are such that, the function

$$s(t) = \sum_{n=2}^{\infty} \frac{s_n}{n + \delta_n(\pi, \beta)} \cos(n + \delta_n(\pi, \beta)) t \quad (0.22)$$

is absolutely continuous on $[0, 2\pi]$ (we will write $s \in AC[0, 2\pi]$).

To prove this theorem, which solves inverse Sturm-Liouville problem $L(q, \pi, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and $\beta \in (0, \pi)$, and provide an efficient algorithm for the reconstruction we solve several problems, each of them represents separate interest:

- 1) In Section 1.1 of Chapter 1 uniform convergence of the expansion of an absolutely continuous function for eigenfunctions of the Sturm-Liouville problems $L(q, \pi, \beta)$, $\beta \in (0, \pi)$ and $L(q, \alpha, 0)$, $\alpha \in (0, \pi)$, with summable potential $q \in L^1_{\mathbb{R}}[0, \pi]$ is proved;
- 2) In Section 1.2 we prove more precise asymptotic formula for the eigenvalues of the problem $L(q, \pi, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and $\beta \in (0, \pi)$;
- 3) In Section 1.3 we derive new and more precise asymptotic formulae for the norming constants of the problem $L(q, \alpha, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and $(\alpha, \beta) \in (0, \pi] \times [0, \pi)$;
- 4) In Section 1.4 we investigate the Riesz basicity of the systems $\{\cos \lambda_n x\}_{n=0}^{\infty}$ and $\{\sin \lambda_n x\}_{n=0}^{\infty}$ in $L^2[0, \pi]$;
- 5) In Section 2.1 of Chapter 2 we derive an analogue of the Gelfand-Levitan equation for our case ($\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0, \pi]$);
- 6) In Section 2.2 the existence and uniqueness of the solution of this Gelfand-Levitan equation as well as reconstruction of the function q (i.e. the reconstruction of differential equation (0.4)) and parameter $\tilde{\beta}$ (see Remark 2.2) are given.

- 7) In Section 2.3 we provide an example of the implementation of our algorithm (by taking two certain sequences and using reconstruction technique we find the potential q and the parameter $\tilde{\beta}$).
- 8) In Section 2.4 we prove three auxiliary lemmas, which are used to obtain some results of this work.

Besides, in Section 3.1 of Chapter 3 we study the dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters that define the boundary conditions. As a corollary, we obtain Sturm oscillation theorem, which states that the n -th eigenfunction has n zeros.

The thesis is exposed on 85 pages. It is comprised of Introduction, 3 Chapters divided into 9 Sections, Conclusion, Acknowledgments, Bibliography, consisting of 94 items, and the authors publications on the topic of the thesis (see [AP1–AP11]).

Asymptotics, Basicity and Expansion

In this chapter we investigate the uniform convergence of the expansion of an absolutely continuous function for eigenfunctions of the Sturm-Liouville problems (see Section 1.1). This result is used to obtain new, more precise asymptotic formulae for eigenvalues (see Section 1.2) and norming constants (see Section 1.3). In this chapter we also study the basicity properties of the systems $\{\cos \lambda_n x\}_{n=0}^{\infty}$ and $\{\sin \lambda_n x\}_{n=0}^{\infty}$ ($\lambda_n^2 \equiv \mu_n$, $n = 0, 1, 2, \dots$) in $L^2[0, \pi]$ (see Section 1.4).

1.1 Expansion Theorems

Completeness and expansion theorems for eigenfunctions of the Sturm-Liouville boundary-value problem have been proven since XIX-th century (see [22, 23, 40–43]). One of the main theorems of the spectral theory of differential operators is as follows (see [36]):

Theorem 1.1 ([36, p. 90]) *Every function in the domain of self-adjoint differential operator can be expanded as a uniformly convergent generalized Fourier series for eigenfunctions of this operator.*

This result cannot be applied for the functions which do not belong to the domain of self-adjoint differential operator. On the other hand, it was proved that absolute continuity of the function f on $[0, \pi]$ is sufficient for Fourier series for the eigenfunctions of the Sturm-Liouville operator $L(q, \alpha, \beta)$, $q \in L^2_{\mathbb{R}}[0, \pi]$, $\alpha, \beta \in (0, \pi)$ to converge uniformly to f (see [35, 39, 47, 62]):

Theorem 1.2 ([62]) *Let $q \in L^2_{\mathbb{R}}[0, \pi]$, $\alpha, \beta \in (0, \pi)$ and f be an absolutely continuous function*

on $[0, \pi]$. Then

$$\lim_{N \rightarrow \infty} \max_{x \in [0, \pi]} \left| f(x) - \sum_{n=0}^N c_n \varphi_n(x) \right| = 0, \quad c_n = \frac{1}{a_n} \int_0^\pi f(t) \varphi_n(t) dt,$$

where $\varphi_n(x) \equiv \varphi(x, \mu_n(q, \alpha, \beta), \alpha)$.

We prove that the analogous results for the problems $L(q, \pi, \beta)$, $\beta \in (0, \pi)$ and $L(q, \alpha, 0)$, $\alpha \in (0, \pi)$ are also true for more general potentials $q \in L^1_{\mathbb{R}}[0, \pi]$:

Theorem 1.3 *Let $q \in L^1_{\mathbb{R}}[0, \pi]$, $\alpha = \pi$, $\beta \in (0, \pi)$ and f be an absolutely continuous function on $[0, \pi]$. Then for arbitrary $a \in (0, \pi)$*

$$\lim_{N \rightarrow \infty} \max_{x \in [a, \pi]} \left| f(x) - \sum_{n=0}^N c_n \varphi_n(x) \right| = 0, \quad c_n = \frac{1}{a_n} \int_0^\pi f(t) \varphi_n(t) dt, \quad (1.1)$$

where $\varphi_n(x) \equiv \varphi(x, \mu_n(q, \pi, \beta), \pi) \equiv \varphi(x, \mu_n, \pi)$.

Theorem 1.4 *Let $q \in L^1_{\mathbb{R}}[0, \pi]$, $\alpha \in (0, \pi)$, $\beta = 0$ and f be an absolutely continuous function on $[0, \pi]$. Then for arbitrary $b \in (0, \pi)$*

$$\lim_{N \rightarrow \infty} \max_{x \in [0, b]} \left| f(x) - \sum_{n=0}^N c_n \varphi_n(x) \right| = 0, \quad c_n = \frac{1}{a_n} \int_0^\pi f(t) \varphi_n(t) dt, \quad (1.2)$$

where $\varphi_n(x) \equiv \varphi(x, \mu_n(q, \alpha, 0), \alpha)$.

We provide the proof for Theorem 1.3. Theorem 1.4 can be proved similarly.

Proof. For $|\lambda| \rightarrow \infty$, the following asymptotic formulae hold ([36, 38, 48, 70–72])

$$\varphi(x, \mu, \pi) := \varphi_\pi(x, \mu) \equiv \varphi_\pi(x, \lambda^2) = \frac{\sin \lambda x}{\lambda} + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^2}\right), \quad (1.3)$$

$$\varphi'(x, \mu, \pi) := \varphi'_\pi(x, \mu) \equiv \varphi'_\pi(x, \lambda^2) = \cos \lambda x + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|}\right), \quad (1.4)$$

$$\begin{aligned} \psi(x, \mu, \beta) := \psi_\beta(x, \mu) \equiv \psi_\beta(x, \lambda^2) &= \cos \lambda(\pi - x) \sin \beta + \frac{\sin \lambda(\pi - x)}{\lambda} \cos \beta + \\ &+ O\left(\frac{e^{|Im\lambda|(\pi-x)}}{|\lambda|}\right) \sin \beta + O\left(\frac{e^{|Im\lambda|(\pi-x)}}{|\lambda|^2}\right) \cos \beta, \end{aligned} \quad (1.5)$$

$$\begin{aligned} \psi'(x, \mu, \beta) := \psi'_\beta(x, \mu) \equiv \psi'_\beta(x, \lambda^2) &= (\lambda \sin \lambda(\pi - x) + O(e^{|Im\lambda|(\pi-x)})) \sin \beta - \\ &- \left(\cos \lambda(\pi - x) + O\left(\frac{e^{|Im\lambda|(\pi-x)}}{|\lambda|}\right) \right) \cos \beta. \end{aligned} \quad (1.6)$$

From (0.9), (0.10) and (1.5) for Wronskian $W_{\pi,\beta}(\mu)$ we have the following estimates

$$W_{\pi,\beta}(\mu) \equiv W_{\pi,\beta}(\lambda^2) = -\psi_\beta(0, \mu) = -\cos \lambda\pi \sin \beta - \frac{\sin \lambda\pi}{\lambda} \cos \beta + \\ + O\left(\frac{e^{|\operatorname{Im}\lambda|\pi}}{|\lambda|}\right) \sin \beta + O\left(\frac{e^{|\operatorname{Im}\lambda|\pi}}{|\lambda|^2}\right) \cos \beta. \quad (1.7)$$

Denote by $\mathbb{Z}_{1/6}$ the following domain of the complex plane \mathbb{C} :

$$\mathbb{Z}_{1/6} = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{n}{2} \right| \geq \frac{1}{6}, n \in \mathbb{Z} \right\}.$$

The following lemma have been proven in [73] by the methods, which had been used in [61, Lemma 1 on p. 27] (see, also [37, 62])

Lemma 1.1 ([73]) *If $\lambda \in \mathbb{Z}_{1/6}$, then*

$$|\sin \pi\lambda| \geq \frac{1}{7}e^{|\operatorname{Im}\lambda|\pi}, \quad |\cos \pi\lambda| \geq \frac{1}{7}e^{|\operatorname{Im}\lambda|\pi}. \quad (1.8)$$

It follows from (1.7) and (1.8) that for sufficiently large $\lambda^* > 0$, there is a constant $C_1 > 0$ such that

$$|W_{\pi,\beta}(\lambda^2)| \geq C_1 e^{|\operatorname{Im}\lambda|\pi} \sin \beta, \quad \text{when } \lambda \in \mathbb{Z}_{1/6}, |\lambda| > \lambda^*. \quad (1.9)$$

Let us consider the following boundary value problem

$$-y'' + q(x)y = \mu y - f(x), \quad x \in (0, \pi), \quad \mu \in \mathbb{C}, \quad f \in L^1[0, \pi], \quad (1.10)$$

$$y(0) = 0, \quad y(\pi) \cos \beta + y'(\pi) \sin \beta = 0, \quad \beta \in (0, \pi). \quad (1.11)$$

It is well-known and can be easily verified that the solution $y(x, \mu, f)$ of the boundary value problem (1.10)–(1.11) can be written in the following form for all $\mu \in \mathbb{C} \setminus \{\mu_n\}_{n=0}^\infty$ (see, for example, [36, 47])

$$y(x, \mu, f) = \frac{1}{W_{\pi,\beta}(\mu)} \psi_\beta(x, \mu) \int_0^x f(t) \varphi_\pi(t, \mu) dt + \\ + \frac{1}{W_{\pi,\beta}(\mu)} \varphi_\pi(x, \mu) \int_x^\pi f(t) \psi_\beta(t, \mu) dt. \quad (1.12)$$

Since φ , ψ and $W_{\pi,\beta}$ are entire functions of μ , then we see that $y(x, \mu, f)$ is a meromorphic function of μ , with poles in the zeros of $W_{\pi,\beta}$ or, that is the same, in eigenvalues $\mu_n, n =$

$0, 1, 2, \dots$. Since $\dot{W}_{\pi, \beta}(\mu_n) \equiv \frac{d}{d\mu} W_{\pi, \beta}(\mu_n) = \beta_n a_n$ (see [62, Lemma 1.1.1]), then using (0.11), we get the residue

$$\begin{aligned} \operatorname{Res}_{\mu=\mu_n} y(x, \mu, f) &= \frac{1}{\dot{W}_{\pi, \beta}(\mu_n)} \psi_\beta(x, \mu_n) \int_0^x f(t) \varphi_\pi(t, \mu_n) dt + \\ &+ \frac{1}{\dot{W}_{\pi, \beta}(\mu_n)} \varphi_\pi(x, \mu_n) \int_x^\pi f(t) \psi_\beta(t, \mu_n) dt = \frac{\beta_n}{\dot{W}_{\pi, \beta}(\mu_n)} \varphi_\pi(x, \mu_n) \int_0^\pi f(t) \varphi_\pi(t, \mu_n) dt = \\ &= \frac{1}{a_n} \varphi_\pi(x, \mu_n) \int_0^\pi f(t) \varphi_\pi(t, \mu_n) dt. \end{aligned} \quad (1.13)$$

It follows from (1.3), (1.5), (1.9) and (1.12) that there are positive numbers C, C_2, C_3, C_4 such that the following estimates hold for $\lambda \in \mathbb{Z}_{1/6}, |\lambda| > \lambda^*$:

$$\begin{aligned} |y(x, \lambda^2, f)| &\leq \frac{|\psi_\beta(x, \lambda^2)| \max_{t \in [0, x]} |\varphi_\pi(t, \lambda^2)| \int_0^x |f(t)| dt}{C_1 e^{|\operatorname{Im} \lambda| \pi} \sin \beta} + \\ &+ \frac{|\varphi_\pi(x, \lambda^2)| \max_{t \in [x, \pi]} |\psi_\beta(t, \lambda^2)| \int_x^\pi |f(t)| dt}{C_1 e^{|\operatorname{Im} \lambda| \pi} \sin \beta} \leq \\ &\leq \frac{e^{|\operatorname{Im} \lambda|(\pi-x)} \left(\sin \beta + \frac{|\cos \beta|}{|\lambda|} + C_3 \frac{\sin \beta}{|\lambda|} + C_4 \frac{|\cos \beta|}{|\lambda|^2} \right)}{C_1 e^{|\operatorname{Im} \lambda| \pi} \sin \beta} e^{|\operatorname{Im} \lambda| x} \left(\frac{1}{|\lambda|} + C_2 \frac{1}{|\lambda|^2} \right) \int_0^\pi |f(t)| dt \leq \\ &\leq \frac{1}{C_1} \int_0^\pi |f(t)| dt \left(\frac{1}{|\lambda|} + O\left(\frac{1}{|\lambda|^2}\right) \right) \leq \frac{C}{|\lambda|}. \end{aligned} \quad (1.14)$$

Let us now consider a function $f \in AC[0, \pi]$. Using the fact that $\varphi_\pi(x, \mu)$ and $\psi_\beta(x, \mu)$ are the solutions of (0.4), we can rewrite the representation (1.12) for $y(x, \mu, f)$ in the following form (compare with [62]):

$$y(x, \mu, f) = \frac{f(x)}{\mu} + f(0) \frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)} + \frac{Z_1(x, \mu, \pi, \beta, f')}{\mu} + \frac{Z_2(x, \mu, \pi, \beta)}{\mu}, \quad (1.15)$$

where

$$Z_1(x, \mu, \pi, \beta, f') = \frac{\psi_\beta(x, \mu) \int_0^x f'(t) \varphi'_\pi(t, \mu) dt + \varphi_\pi(x, \mu) \int_x^\pi f'(t) \psi'_\beta(t, \mu) dt}{W_{\pi, \beta}(\mu)}, \quad (1.16)$$

$$\begin{aligned} Z_2(x, \mu, \pi, \beta) &= -f(\pi) \psi'_\beta(\pi, \mu) \frac{\varphi_\pi(x, \mu)}{W_{\pi, \beta}(\mu)} + y(x, \mu, qf) = \\ &= f(\pi) \cos \beta \frac{\varphi_\pi(x, \mu)}{W_{\pi, \beta}(\mu)} + y(x, \mu, qf). \end{aligned} \quad (1.17)$$

Let us show that

$$\lim_{\substack{|\lambda| \rightarrow \infty \\ \lambda \in \mathbb{Z}_{1/6}}} \max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta, f')| = 0. \quad (1.18)$$

First, we suppose that f' is absolutely continuous function on $[0, \pi]$. Then there exists $f'' \in L^1[0, \pi]$ and (1.16) can be written in the following form

$$\begin{aligned} Z_1(x, \mu, \pi, \beta, f') &= \frac{\psi_\beta(x, \mu)}{W_{\pi, \beta}(\mu)} \left(\varphi_\pi(t, \mu) f'(t) \Big|_{t=0}^x - \int_0^x f''(t) \varphi_\pi(t, \mu) dt \right) + \\ &+ \frac{\varphi_\pi(x, \mu)}{W_{\pi, \beta}(\mu)} \left(\psi_\beta(t, \mu) f'(t) \Big|_{t=x}^\pi - \int_x^\pi f''(t) \psi_\beta(t, \mu) dt \right) = \frac{\varphi_\pi(x, \mu)}{W_{\pi, \beta}(\mu)} f'(\pi) \sin \beta - \\ &\quad \frac{\psi_\beta(x, \mu) \int_0^x f''(t) \varphi_\pi(t, \mu) dt + \varphi_\pi(x, \mu) \int_x^\pi f''(t) \psi_\beta(t, \mu) dt}{W_{\pi, \beta}(\mu)}. \end{aligned}$$

By virtue of (1.3)–(1.6) and (1.9) we obtain that there is a number $C > 0$, such that

$$\max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta, f')| \leq \frac{C}{|\lambda|}, \text{ when } \lambda \in \mathbb{Z}_{1/6}, |\lambda| > \lambda^*.$$

It implies (1.18) in the case $f' \in AC[0, \pi]$.

Now, let us turn to the general case $g := f' \in L^1[0, \pi]$. Fix $\epsilon > 0$ and choose an absolutely continuous function g_ϵ , such that

$$\int_0^\pi |g(t) - g_\epsilon(t)| dt < \frac{C_1 \sin \beta}{16} \epsilon.$$

Then, according to (1.3)–(1.6), (1.9) and (1.16) for $\lambda \in \mathbb{Z}_{1/6}$, $|\lambda| > \lambda^*$, we have

$$\begin{aligned} \max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta, g)| &\leq \max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta, g_\epsilon)| + \max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta, g - g_\epsilon)| \leq \\ &\leq \frac{C(\epsilon)}{|\lambda|} + \frac{C_1 \sin \beta}{16} \epsilon \max_{x \in [0, \pi]} \left(\frac{|\psi_\beta(x, \mu)| \max_{t \in [0, x]} |\varphi'_\pi(t, \mu)| + |\varphi_\pi(x, \mu)| \max_{t \in [0, x]} |\psi'_\beta(t, \mu)|}{C_1 e^{Im\lambda\pi} \sin \beta} \right) \leq \\ &\leq \frac{C(\epsilon)}{|\lambda|} + \frac{C_1 \sin \beta}{16} \epsilon \max_{x \in [0, \pi]} \left(\frac{8e^{Im\lambda\pi}}{C_1 e^{Im\lambda\pi} \sin \beta} \right) \leq \frac{C(\epsilon)}{|\lambda|} + \frac{\epsilon}{2}. \end{aligned}$$

It is easy to see, that if we choose $\lambda_\epsilon^* = \frac{2C(\epsilon)}{\epsilon}$, then for $\lambda \in \mathbb{Z}_{1/6}$, $|\lambda| > \lambda_\epsilon^*$ we have

$\max_{x \in [0, \pi]} |Z_1(x, \mu, \pi, \beta)| \leq \epsilon$. Due to the arbitrariness $\epsilon > 0$, we arrive at (1.18).

Now, we estimate $Z_2(x, \mu, \pi, \beta)$ (see (1.17)). Since $qf \in L^1[0, \pi]$, then the estimates in (1.14) are also true for $y(x, \mu, qf)$. Using (1.3), (1.9), (1.14) and the fact that $\sin \beta \neq 0$ we get

the following estimates for $\lambda \in \mathbb{Z}_{1/6}$, $|\lambda| > \lambda^*$

$$\begin{aligned} \max_{x \in [0, \pi]} |Z_2(x, \mu, \pi, \beta)| &\leq \max_{x \in [0, \pi]} \left| f(\pi) \cos \beta \frac{\varphi_\pi(x, \mu)}{W_{\pi, \beta}(\mu)} \right| + \max_{x \in [0, \pi]} |y(x, \mu, qf)| \leq \\ &\leq \left| f(\pi) \cos \beta \frac{C_5 e^{Im\lambda|\pi}}{|\lambda| C_1 e^{Im\lambda|\pi} \sin \beta} \right| + \frac{C_6}{|\lambda|} \leq \frac{C_5}{C_1} \frac{|f(\pi) \cot \beta|}{|\lambda|} + \frac{C_6}{|\lambda|} \leq \frac{C_7}{|\lambda|}, \end{aligned} \quad (1.19)$$

where C_5, C_6, C_7 are positive numbers.

Consider the following contour integral

$$I_N(x) = \frac{1}{2\pi i} \oint_{\Gamma_N} y(x, \mu, f) d\mu, \quad (1.20)$$

where $\Gamma_N = \left\{ \mu : |\mu| = \left(N + \frac{3}{4} \right)^2 \right\}$ (with counterclockwise circuit). On one hand, using the Cauchy's residue theorem (see [74]), from (1.13) we get

$$I_N(x) = \sum_{n=0}^N \frac{1}{a_n} \int_0^\pi f(t) \varphi_\pi(t, \mu_n) dt \varphi_\pi(x, \mu_n). \quad (1.21)$$

On the other hand, from (1.15), (1.18) and (1.19) we obtain

$$I_N(x) = f(x) + f(0) \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)} d\mu + \epsilon_N(x), \quad (1.22)$$

where $\epsilon_N(x)$, according to (1.18) and (1.19), uniformly converges to 0 :

$$\lim_{N \rightarrow \infty} \max_{x \in [0, \pi]} |\epsilon_N(x)| = 0.$$

Without loss of generality we assume that $\mu = 0$ is not an eigenvalue of the problem $L(q, \pi, \beta)$.

Indeed, if 0 is an eigenvalue, then there is a number c such that $\mu_n + c \neq 0$, $n = 0, 1, 2, \dots$ are the eigenvalues of the problem $L(q + c, \pi, \beta)$ with the same eigenfunctions φ_n and norming constants a_n as for $L(q, \pi, \beta)$. Then the function $\frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)}$ has only first order poles and using Cauchy's residue theorem (see [74]), we can easily calculate that

$$\begin{aligned} \phi_N(x) &:= \frac{1}{2\pi i} \oint_{\Gamma_N} \frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)} d\mu = \operatorname{Res}_{\mu=0} \frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)} + \sum_{n=0}^N \operatorname{Res}_{\mu=\mu_n} \frac{\psi_\beta(x, \mu)}{\mu W_{\pi, \beta}(\mu)} = \\ &= \frac{\psi_\beta(x, 0)}{W_{\pi, \beta}(0)} + \sum_{n=0}^N \frac{\psi_\beta(x, \mu_n)}{\mu_n \dot{W}_{\pi, \beta}(\mu_n)} = \frac{\psi_\beta(x, 0)}{W_{\pi, \beta}(0)} + \sum_{n=0}^N \frac{\beta_n \varphi_\pi(x, \mu_n)}{\mu_n \beta_n a_n} = \\ &= \frac{\psi_\beta(x, 0)}{W_{\pi, \beta}(0)} + \sum_{n=0}^N \frac{1}{\mu_n a_n} \varphi_n(x). \end{aligned} \quad (1.23)$$

Now let us show that the sequence $\phi_N(x)$ converges to 0 (when $N \rightarrow \infty$) uniformly on segment $[a, \pi]$, for arbitrary $a \in (0, \pi)$.

Since $\varphi_n(x) = \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} + O\left(\frac{1}{n^2}\right)$ uniformly on $[0, \pi]$ (see (1.3)), $\mu_n = \mu_n(q, \pi, \beta) = \left(n + \frac{1}{2}\right)^2 + O(1)$ (see [69, Theorem 1 on p. 286 and asymptotical estimate for $\delta_n(\pi, \beta)$ on p. 292]) and $a_n = a_n(q, \pi, \beta) = \frac{\pi}{2(n + \frac{1}{2})^2} \left(1 + o\left(\frac{1}{n}\right)\right)$ (see Theorem 1.8 on p. 27), then $\phi_N(x)$ (see (1.23)) can be written in the following form:

$$\phi_N(x) = \frac{\psi_\beta(x, 0)}{W_{\pi, \beta}(0)} + \frac{2}{\pi} \sum_{n=0}^N \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} + \sum_{n=0}^N q_n(x),$$

where $q_n(x) = O\left(\frac{1}{n^2}\right)$ uniformly on $[0, \pi]$.

Since $\sum_{n=0}^{\infty} \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} = \frac{\pi}{2}$, $0 < x < 2\pi$ (see, for example, [75, formula 37 on p. 578]), then the sequence $\phi_N(x)$ converges to the continuous function $\phi(x)$ (when $N \rightarrow \infty$) uniformly on segment $[a, \pi]$, for arbitrary $a \in (0, \pi)$.

Now, to prove that $\phi(x) \equiv 0$, $x \in (0, \pi]$, it is sufficient to prove that $\phi = 0$ in $L^2(0, \pi)$.

To this aim, by doing some calculations we get:

$$\int_0^\pi \phi(x) \varphi_m(x) dx = \frac{1}{W_{\pi, \beta}(0)} \int_0^\pi \psi_\beta(x, 0) \varphi_m(x) dx + \frac{1}{\mu_m}, \quad m = 0, 1, 2, \dots \quad (1.24)$$

and

$$\begin{aligned} \mu_m \int_0^\pi \psi_\beta(x, 0) \varphi_m(x) dx &= \int_0^\pi (\varphi_m(x) \psi_\beta''(x, 0) - \varphi_m''(x) \psi_\beta(x, 0)) dx = \\ &= (\varphi_m(x) \psi_\beta'(x, 0) - \varphi_m'(x) \psi_\beta(x, 0)) \Big|_0^\pi = \psi_\beta(0, 0) = -W_{\pi, \beta}(0). \end{aligned} \quad (1.25)$$

It follows from (1.24) and (1.25) that

$$\int_0^\pi \phi(x) \varphi_m(x) dx = 0, \quad m = 0, 1, 2, \dots$$

Since the system of eigenfunctions $\{\varphi_m(x)\}_{m=0}^\infty$ of the boundary value problem $L(q, \pi, \beta)$ is complete and orthogonal in $L^2(0, \pi)$, then $\phi = 0$ in $L^2(0, \pi)$.

Comparing this result with (1.21), (1.22) and passing to the limit ($N \rightarrow \infty$) we arrive at (1.1). Theorem 1.3 is proved. \square

The attentive reader might offer the equiconvergence argument to obtain the result. Below in remark 1.1 we briefly explain why this argument can't be applied in our case.

Remark 1.1 *It is well known that one of the proofs of the classical Theorem 1.2 is based on the so-called uniformly equiconvergence theorem, which states that the expansion in eigenfunctions of the problem $L(q, \alpha, \beta)$, $\alpha, \beta \in (0, \pi)$ is equivalent to the expansion in eigenfunctions of the problem $L\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$, i.e., $\{\cos nx\}_{n \geq 0}$ (see [35, 39, 47]). Further, the Dirichlet-Jordan theorem (see [76, pp. 121–122]) can be applied and Theorem 1.2 will be proved. On one hand, it is easy to set up a similar equiconvergence assertion for expansion in eigenfunctions of the problem $L(q, \pi, \beta)$, $\beta \in (0, \pi)$. This is equivalent to the expansion in $\sin\left(n + \frac{1}{2}\right)x$, $n = 0, 1, 2, \dots$, i.e., eigenfunctions of the problem $L\left(0, \pi, \frac{\pi}{2}\right)$ (see [39, remark on p. 304] and [35, remark on p. 71]). On the other hand, to the best of our knowledge, there are no analogous result of the Dirichlet-Jordan theorem for expansion in the system of functions $\left\{\sin\left(n + \frac{1}{2}\right)x\right\}_{n \geq 0}$ (for the nearest result see [77, Theorem 2.6]). To overcome this difficulty, we have proved Theorem 1.3 using the Cauchy's contour integral method.*

Remark 1.2 *It is easy to see that if we demand $f(0) = 0$ in Theorem 1.3 then the series in (1.1) converges uniformly on whole segment $[0, \pi]$. The same is true for Theorem 1.4 if we demand $f(\pi) = 0$.*

1.2 Asymptotics of the Eigenvalues

As it was mentioned above, in [69] Harutyunyan introduced the function $\delta_n(\alpha, \beta)$ (see (0.17) and (0.18)). In the same paper the following theorem was proved:

Theorem 1.5 ([69]) *Let $q \in L^1_{\mathbb{R}}[0, \pi]$. The lowest eigenvalue has the properties:*

$$\lim_{\alpha \rightarrow 0} \mu_0(q, \alpha, \beta) = -\infty, \quad \lim_{\beta \rightarrow \pi} \mu_0(q, \alpha, \beta) = -\infty. \quad (1.26)$$

For eigenvalues $\mu_n(q, \alpha, \beta)$, $n \geq 2$, the following formula

$$\mu_n(q, \alpha, \beta) = (n + \delta_n(\alpha, \beta))^2 + [q] + r_n(q, \alpha, \beta) \quad (1.27)$$

holds, where $[q] = \frac{1}{\pi} \int_0^\pi q(x) dx$ and $r_n(q, \alpha, \beta) = o(1)$, when $n \rightarrow \infty$, uniformly in $\alpha, \beta \in [0, \pi]$ and $q \in BL^1_{\mathbb{R}}[0, \pi]$ (here and below $BL^1_{\mathbb{R}}[0, \pi]$ stands for bounded subsets of $L^1_{\mathbb{R}}[0, \pi]$).

It is worthwhile to mention that prior to [69] the dependence of spectral data (by spectral data here we understand the set of eigenvalues and the set of norming constants) on α and β has been usually studied (see [35–37, 47, 48, 55, 61–64]) in the following sense: the boundary conditions are separated into four cases and results are formulated separately for each case (for details see [48, p. 386]), namely:

- 1) $\mu_n(q, \alpha, \beta) = n^2 + \frac{2}{\pi} (\cot \beta - \cot \alpha) + [q] + r_n(q, \alpha, \beta)$, if $\alpha, \beta \in (0, \pi)$,
- 2) $\mu_n(q, \pi, \beta) = \left(n + \frac{1}{2}\right)^2 + \frac{2}{\pi} \cot \beta + [q] + r_n(q, \beta)$, if $\beta \in (0, \pi)$,
- 3) $\mu_n(q, \alpha, 0) = \left(n + \frac{1}{2}\right)^2 - \frac{2}{\pi} \cot \alpha + [q] + r_n(q, \alpha)$, if $\alpha \in (0, \pi)$,
- 4) $\mu_n(q, \pi, 0) = (n + 1)^2 + [q] + r_n(q)$,

where $r_n = o(1)$ as $n \rightarrow \infty$, but this estimate for r_n is not uniform in $\alpha, \beta \in [0, \pi]$.

The formula (1.27) not only unifies all these formulae into the one, but also ensures uniform convergence of the remainders in accordance with the Theorem 1.5.

In the paper [72] Harutyunyan obtained more precise estimate of the remainder term in the asymptotic formula for eigenvalues. More explicitly, the following theorem has been proved:

Theorem 1.6 ([72]) *Let $q \in L^1_{\mathbb{R}}[0, \pi]$ and let $\lambda_n^2(q, \alpha, \beta) = \mu_n(q, \alpha, \beta)$. Then*

(a) *The asymptotic relation ($n \rightarrow \infty$)*

$$\lambda_n(q, \alpha, \beta) = n + \delta_n(\alpha, \beta) + \frac{[q]}{2(n + \delta_n(\alpha, \beta))} + l_n(q, \alpha, \beta) + O\left(\frac{1}{n^2}\right), \quad (1.28)$$

holds, where $[q] = \frac{1}{\pi} \int_0^\pi q(t) dt$,

$$l_n(q, \alpha, \beta) = \frac{1}{2\pi(n + \delta_n(\alpha, \beta))} \int_0^\pi q(x) \cos 2(n + \delta_n(\alpha, \beta)) x dx, \quad \alpha \in (0, \pi),$$

and

$$l_n = l_n(q, \pi, \beta) = -\frac{1}{2\pi(n + \delta_n(\pi, \beta))} \int_0^\pi q(x) \cos 2(n + \delta_n(\pi, \beta)) x dx. \quad (1.29)$$

The estimate $O\left(\frac{1}{n^2}\right)$ of the remainder in (1.28) is uniform in all $\alpha, \beta \in [0, \pi]$, and $q \in BL^1_{\mathbb{R}}[0, \pi]$.

(b) For $\alpha, \beta \in (0, \pi)$ and for the case $\alpha = \pi, \beta = 0$ the function l , defined by the formula

$$l(x) = \sum_{n=2}^{\infty} l_n(q, \alpha, \beta) \sin(n + \delta_n(\alpha, \beta))x, \quad (1.30)$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, that is $l \in AC(0, 2\pi)$.

The proof of the Theorem 1.6 does not cover the case $\alpha = \pi, \beta \in (0, \pi)$. In the forthcoming theorem using Theorem 1.3 we handle this case as well.

Theorem 1.7 *The function l , defined by the formula*

$$l(x, \beta) = \sum_{n=2}^{\infty} l_n(q, \pi, \beta) \sin(n + \delta_n(\pi, \beta))x \quad (1.31)$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, i.e. $l \in AC(0, 2\pi)$.

Proof. Denote $\sigma(x) = \int_0^x q(t) dt$ and write $l_n(q, \pi, \beta)$ in the following form:

$$\begin{aligned} l_n(q, \pi, \beta) &= -\frac{1}{2\pi(n + \delta_n(\pi, \beta))} \int_0^\pi q(x) \cos 2(n + \delta_n(\pi, \beta))x dx = \\ &= -\frac{1}{2\pi(n + \delta_n(\pi, \beta))} \int_0^\pi \cos 2(n + \delta_n(\pi, \beta))x d\sigma(x) = -\frac{\sigma(\pi) \cos 2\pi\delta_n(\pi, \beta)}{2\pi(n + \delta_n(\pi, \beta))} - \\ &\quad - \frac{1}{\pi} \int_0^\pi \sigma(x) \sin 2(n + \delta_n(\pi, \beta))x dx = -\frac{\sigma(\pi) \cos 2\pi\delta_n(\pi, \beta)}{2\pi(n + \delta_n(\pi, \beta))} - \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \sigma_1(x) \sin(n + \delta_n(\pi, \beta))x dx, \end{aligned} \quad (1.32)$$

where $\sigma_1(x) \equiv \sigma\left(\frac{x}{2}\right)$ is the absolutely continuous function on $[0, 2\pi]$.

It is easy to see from (0.17) and (0.18) (for the details see [69]), that for $\beta \in (0, \pi)$ we have

$$\delta_n(\pi, \beta) = \frac{1}{2} + \frac{\cot \beta}{\pi(n + \frac{1}{2})} + O\left(\frac{1}{n^2}\right) \cot \beta = \frac{1}{2} + O\left(\frac{1}{n}\right), \quad (1.33)$$

and consequently,

$$\cos 2\pi\delta_n(\pi, \beta) = -1 + d_n, \quad \sin 2\pi\delta_n(\pi, \beta) = e_n, \quad (1.34)$$

where $d_n = O\left(\frac{1}{n^2}\right)$, $e_n = O\left(\frac{1}{n}\right)$.

Therefore, $l(x, \beta)$ can be represented as a sum of three functions

$$l(x, \beta) = l_1(x, \beta) + l_2(x, \beta) + l_3(x, \beta), \quad (1.35)$$

where

$$l_1(x, \beta) = \frac{\sigma(\pi)}{2\pi} \sum_{n=2}^{\infty} \frac{\sin(n + \delta_n(\pi, \beta))x}{(n + \delta_n(\pi, \beta))}, \quad (1.36)$$

$$l_2(x, \beta) = -\frac{\sigma(\pi)}{2\pi} \sum_{n=2}^{\infty} d_n \frac{\sin(n + \delta_n(\pi, \beta))x}{(n + \delta_n(\pi, \beta))}, \quad (1.37)$$

$$l_3(x, \beta) = -\frac{1}{2\pi} \sum_{n=2}^{\infty} f_n \sin(n + \delta_n(\pi, \beta))x, \quad (1.38)$$

and $f_n = \int_0^{2\pi} \sigma_1(t) \sin(n + \delta_n(\pi, \beta)) t dt$.

Since $f_n = \int_0^{\pi} \sigma_1(t) \sin(n + \delta_n(\pi, \beta)) t dt + \int_{\pi}^{2\pi} \sigma_1(t) \sin(n + \delta_n(\pi, \beta)) t dt$ and

$$\begin{aligned} \int_{\pi}^{2\pi} \sigma_1(t) \sin(n + \delta_n(\pi, \beta)) t dt &= \int_{-2\pi}^{-\pi} -\sigma_1(-t) \sin(n + \delta_n(\pi, \beta)) t dt = \\ &= \int_0^{\pi} -\sigma_1(2\pi - t) \sin(n + \delta_n(\pi, \beta)) (t - 2\pi) dt = \\ &= \int_0^{\pi} \sigma_1(2\pi - t) ((1 - d_n) \sin(n + \delta_n(\pi, \beta)) t + e_n \cos(n + \delta_n(\pi, \beta)) t) dt = \\ &= \int_0^{\pi} \sigma \left(\pi - \frac{t}{2} \right) ((1 - d_n) \sin(n + \delta_n(\pi, \beta)) t + e_n \cos(n + \delta_n(\pi, \beta)) t) dt, \end{aligned}$$

then

$$\begin{aligned} f_n &= \int_0^{\pi} \left(\sigma \left(\frac{t}{2} \right) + \sigma \left(\pi - \frac{t}{2} \right) \right) \sin(n + \delta_n(\pi, \beta)) t dt - \\ &\quad - d_n \int_0^{\pi} \sigma \left(\pi - \frac{t}{2} \right) \sin(n + \delta_n(\pi, \beta)) t dt + \\ &\quad + e_n \int_0^{\pi} \sigma \left(\pi - \frac{t}{2} \right) \cos(n + \delta_n(\pi, \beta)) t dt. \quad (1.39) \end{aligned}$$

It is noteworthy, that $\delta_n(\alpha, \beta)$ are defined only for $n \geq 2$, that is why, we will write $\lambda_0(0, \pi, \beta)$, $\lambda_1(0, \pi, \beta)$ and $\lambda_n(0, \pi, \beta) = n + \delta_n(\pi, \beta)$ for all $n \geq 2$. Taking into account that the system

of functions

$$\{\varphi_n(x)\}_{n=0}^{\infty} = \left\{ \frac{\sin \lambda_n(0, \pi, \beta) x}{\lambda_n(0, \pi, \beta)} \right\}_{n=0}^{\infty} = \left\{ \frac{\sin \lambda_n(0, \pi, \beta) x}{\lambda_n(0, \pi, \beta)} \right\}_{n=0}^1 \cup \left\{ \frac{\sin(n + \delta_n(\pi, \beta)) x}{n + \delta_n(\pi, \beta)} \right\}_{n=2}^{\infty}$$

are the eigenfunctions of the problem $L(0, \pi, \beta)$ and using Theorem 1.3, we get

$$\begin{aligned} \sigma\left(\frac{x}{2}\right) + \sigma\left(\pi - \frac{x}{2}\right) &= \sigma_2(x) + \\ &+ \sum_{n=2}^{\infty} \frac{\int_0^{\pi} \left(\sigma\left(\frac{t}{2}\right) + \sigma\left(\pi - \frac{t}{2}\right) \right) \sin(n + \delta_n(\pi, \beta)) t dt}{\int_0^{\pi} \sin^2(n + \delta_n(\pi, \beta)) t dt} \sin(n + \delta_n(\pi, \beta)) x \end{aligned} \quad (1.40)$$

where the series converges uniformly on arbitrary segment $[a, \pi] \subset (0, \pi]$ and

$$\sigma_2(x) := \sum_{n=0}^1 \frac{\int_0^{\pi} \left(\sigma\left(\frac{t}{2}\right) + \sigma\left(\pi - \frac{t}{2}\right) \right) \sin \lambda_n(0, \pi, \beta) t dt}{\int_0^{\pi} \sin^2 \lambda_n(0, \pi, \beta) t dt} \sin \lambda_n(0, \pi, \beta) x$$

Using (1.33) and (1.34), we calculate

$$\int_0^{\pi} \sin^2(n + \delta_n(\pi, \beta)) t dt = \frac{\pi}{2} - \frac{\sin 2\pi(n + \delta_n(\pi, \beta))}{4(n + \delta_n(\pi, \beta))} = \frac{\pi}{2} - \frac{e_n}{4(n + \delta_n(\pi, \beta))}. \quad (1.41)$$

From (1.41), it is easy to see that

$$\frac{1}{\int_0^{\pi} \sin^2(n + \delta_n(\pi, \beta)) t dt} = \frac{2}{\pi} + g_n, \quad (1.42)$$

where $g_n = \frac{2e_n}{\pi(2\pi(n + \delta_n(\pi, \beta)) - e_n)} = O\left(\frac{1}{n^2}\right)$.

Now we can write (1.40) in the form

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2}{\pi} \int_0^{\pi} \left(\sigma\left(\frac{t}{2}\right) + \sigma\left(\pi - \frac{t}{2}\right) \right) \sin(n + \delta_n(\pi, \beta)) t dt \sin(n + \delta_n(\pi, \beta)) x &= \\ = - \sum_{n=2}^{\infty} g_n \int_0^{\pi} \left(\sigma\left(\frac{t}{2}\right) + \sigma\left(\pi - \frac{t}{2}\right) \right) \sin(n + \delta_n(\pi, \beta)) t dt \sin(n + \delta_n(\pi, \beta)) x + \\ &+ \sigma\left(\frac{x}{2}\right) + \sigma\left(\pi - \frac{x}{2}\right) - \sigma_2(x), \end{aligned} \quad (1.43)$$

where the series converge uniformly on arbitrary segment $[a, \pi] \subset (0, \pi]$.

Hence, from (1.38), (1.39), (1.43) we obtain that for arbitrary $x \in (0, \pi]$

$$\begin{aligned}
l_3(x, \beta) &= \frac{1}{4} \left(-\sigma\left(\frac{x}{2}\right) - \sigma\left(\pi - \frac{x}{2}\right) + \sigma_2(x) \right) + \\
&+ \sum_{n=2}^{\infty} \frac{g_n}{4} \int_0^{\pi} \left(\sigma\left(\frac{t}{2}\right) + \sigma\left(\pi - \frac{t}{2}\right) \right) \sin(n + \delta_n(\pi, \beta)) t dt \sin(n + \delta_n(\pi, \beta)) x + \\
&+ \frac{1}{2\pi} \sum_{n=2}^{\infty} d_n \int_0^{\pi} \sigma\left(\pi - \frac{t}{2}\right) \sin(n + \delta_n(\pi, \beta)) t dt \sin(n + \delta_n(\pi, \beta)) x - \\
&- \frac{1}{2\pi} \sum_{n=2}^{\infty} e_n \int_0^{\pi} \sigma\left(\pi - \frac{t}{2}\right) \cos(n + \delta_n(\pi, \beta)) t dt \sin(n + \delta_n(\pi, \beta)) x. \quad (1.44)
\end{aligned}$$

Since $d_n = O\left(\frac{1}{n^2}\right)$, $e_n \int_0^{\pi} \sigma\left(\pi - \frac{t}{2}\right) \cos(n + \delta_n(\pi, \beta)) t dt = O\left(\frac{1}{n^2}\right)$, $g_n = O\left(\frac{1}{n^2}\right)$, then $l_3 \in AC(0, \pi]$. But on the other hand, since (see (1.38) and (1.34))

$$\begin{aligned}
l_3(2\pi - x, \beta) &= l_3(x, \beta) + \frac{1}{2\pi} \sum_{n=2}^{\infty} d_n f_n \sin(n + \delta_n(\pi, \beta)) x - \\
&- \frac{1}{2\pi} \sum_{n=2}^{\infty} e_n f_n \cos(n + \delta_n(\pi, \beta)) x,
\end{aligned}$$

then $l_3 \in AC[\pi, 2\pi)$ and consequently $l_3 \in AC(0, 2\pi)$.

Since $\sum_{n=2}^{\infty} \frac{\sin(n + \delta_n(\pi, \beta)) x}{(n + \delta_n(\pi, \beta))}$ is absolutely continuous function on $(0, 2\pi)$ (see section 2.4, Lemma 2.6), then $l_1 \in AC(0, 2\pi)$.

Since $d_n = O\left(\frac{1}{n^2}\right)$, then the series in (1.37) and its first derivative converges absolutely and uniformly on $[0, 2\pi]$ and therefore $l_2 \in AC[0, 2\pi]$. This completes the proof. \square

1.3 Asymptotics of the Norming Constants

The dependence of norming constants (see (0.12)) on α and β (as far as we know) hasn't been investigated before. So far, for norming constants the following is known.

In the case $\sin \alpha \neq 0$, for absolutely continuous q , it is known (see [35]) that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2 \alpha} = \frac{\pi}{2} + O\left(\frac{1}{n^2}\right). \quad (1.45)$$

For $q(x) = \frac{dF(x)}{dx}$, where F is a bounded variation function on $[0, \pi]$, $\lim_{x \rightarrow 0^+} F(x) = F(0)$ and $\lim_{x \rightarrow \pi^-} F(x) = F(\pi)$, it has been asserted (see Theorem 0.2) that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2 \alpha} = \frac{\pi}{2} + \alpha_n, \quad (1.46)$$

where the sequence $\{\alpha_n\}_{n=0}^\infty$ is characterized by the condition that the function $\sum_{n=0}^\infty \alpha_n \cos nx = \alpha(x) \in BV[0, \pi]$, i.e. has a bounded variation on $[0, \pi]$.

For $q \in L^2_{\mathbb{R}}[0, \pi]$, it has been proved (see Theorem 0.3) that

$$\frac{a_n(q, \alpha, \beta)}{\sin^2 \alpha} = \frac{\pi}{2} + \frac{\kappa_n}{n}, \quad (1.47)$$

where $\{\kappa_n\}_{n=0}^\infty \in l^2$, i.e. $\sum_{n=0}^\infty |\kappa_n|^2 < \infty$.

In the case $\sin \alpha = 0$, $\sin \beta \neq 0$, for $q \in AC[0, \pi]$, it is known (see [35]) that

$$a_n(q, \pi, \beta) = \frac{\pi}{2(n+1/2)^2} \left[1 + O\left(\frac{1}{n^2}\right) \right]. \quad (1.48)$$

We derive new asymptotic formulae for the norming constants of Sturm-Liouville problem with summable potentials, which generalize and make more precise previously known formulae. Moreover, our formulae take into account the smooth dependence of norming constants on boundary conditions.

Theorem 1.8 *Let $q \in L^1_{\mathbb{R}}[0, \pi]$. For norming constants a_n and b_n the following asymptotic formulae hold (when $n \rightarrow \infty$):*

$$\begin{aligned} a_n(q, \alpha, \beta) = & \frac{\pi}{2} \left[1 + \frac{2s_n(q, \alpha, \beta)}{\pi[n + \delta(\alpha, \beta)]} + r_n \right] \sin^2 \alpha + \\ & + \frac{\pi}{2[n + \delta_n(\alpha, \beta)]^2} \left[1 + \frac{2s_n(q, \alpha, \beta)}{\pi[n + \delta(\alpha, \beta)]} + \tilde{r}_n \right] \cos^2 \alpha, \end{aligned} \quad (1.49)$$

$$\begin{aligned} b_n(q, \alpha, \beta) = & \frac{\pi}{2} \left[1 + \frac{2s_n(q, \alpha, \beta)}{\pi[n + \delta(\alpha, \beta)]} + p_n \right] \sin^2 \beta + \\ & + \frac{\pi}{2[n + \delta_n(\alpha, \beta)]^2} \left[1 + \frac{2s_n(q, \alpha, \beta)}{\pi[n + \delta(\alpha, \beta)]} + \tilde{p}_n \right] \cos^2 \beta, \end{aligned}$$

where

$$s_n = s_n(q, \alpha, \beta) = -\frac{1}{2} \int_0^\pi (\pi - t) q(t) \sin 2[n + \delta_n(\alpha, \beta)] t dt, \quad (1.50)$$

$r_n = r_n(q, \alpha, \beta) = O\left(\frac{1}{n^2}\right)$ and $\tilde{r}_n = \tilde{r}_n(q, \alpha, \beta) = O\left(\frac{1}{n^2}\right)$ (the same estimate is true for p_n and \tilde{p}_n), when $n \rightarrow \infty$, uniformly in $\alpha, \beta \in [0, \pi]$ and $q \in BL^1_{\mathbb{R}}[0, \pi]$.

Remark 1.3 *It is important to note that norming constants $a_n(q, \alpha, \beta)$ and $b_n(q, \alpha, \beta)$ are analytic functions on α and β . It easily follows from formulae (1.64), (1.65) below, (1.28) and from the result in [69], which states that $\lambda_n(q, \alpha, \beta)$ ($\lambda_n^2(q, \alpha, \beta) = \mu_n(q, \alpha, \beta)$) depend analytically on α and β .*

Proof. Our proof based on the detailed study of the dependence of eigenfunctions φ_n and ψ_n on parameters α and β .

Let $q \in L^1_{\mathbb{C}}[0, \pi]$, i.e. q is a complex-valued, summable function on $[0, \pi]$, and let us denote by $y_i(x, \lambda)$, $i = 1, 2, 3, 4$, the solutions of the equation

$$-y'' + q(x)y = \lambda^2 y, \quad (1.51)$$

satisfying the initial conditions

$$\begin{aligned} y_1(0, \lambda) = 1, \quad y_2(0, \lambda) = 0, \quad y_3(\pi, \lambda) = 1, \quad y_4(\pi, \lambda) = 0, \\ y'_1(0, \lambda) = 0, \quad y'_2(0, \lambda) = 1, \quad y'_3(\pi, \lambda) = 0, \quad y'_4(\pi, \lambda) = 1. \end{aligned} \quad (1.52)$$

Let us recall that by a solution of (1.51) (which is the same as (0.4)) we understand the function y , such that $y, y' \in AC[0, \pi]$ and which satisfies (1.51) almost everywhere (see [36]).

The solutions y_1 and y_2 (as well as the second pair y_3 and y_4) form a fundamental system of solutions of (0.4), i.e. any solution y of (0.4) can be represented in the form:

$$y(x) = y(0)y_1(x, \lambda) + y'(0)y_2(x, \lambda) = y(\pi)y_3(x, \lambda) + y'(\pi)y_4(x, \lambda). \quad (1.53)$$

The existence and uniqueness of the solutions y_i , $i = 1, 2, 3, 4$ (under the condition $q \in L^1_{\mathbb{C}}[0, \pi]$) were investigated in [36, 38, 48, 70, 71]. The following lemma in some sense extends the results of the mentioned papers related to asymptotics (when $|\lambda| \rightarrow \infty$) of the solutions y_i , $i = 1, 2, 3, 4$.

Lemma 1.2 ([72, 73]) *Let $q \in L^1_{\mathbb{C}}[0, \pi]$. Then for the solutions y_i , $i = 1, 2, 3, 4$, the following representations hold (when $|\lambda| \geq 1$):*

$$y_1(x, \lambda) = \cos \lambda x + \frac{1}{2\lambda} a(x, \lambda), \quad (1.54)$$

$$y_2(x, \lambda) = \frac{\sin \lambda x}{\lambda} - \frac{1}{2\lambda^2} b(x, \lambda), \quad (1.55)$$

$$y_3(x, \lambda) = \cos \lambda (\pi - x) + \frac{1}{2\lambda} c(x, \lambda), \quad (1.56)$$

$$y_4(x, \lambda) = \frac{\sin \lambda (\pi - x)}{\lambda} - \frac{1}{2\lambda^2} d(x, \lambda), \quad (1.57)$$

where a, b, c, d are twice differentiable with respect to x and entire functions with respect to λ , and have the form

$$a(x, \lambda) = \sin \lambda x \int_0^x q(t) dt + \int_0^x q(t) \sin \lambda (x - 2t) dt + R_1(x, \lambda, q), \quad (1.58)$$

$$b(x, \lambda) = \cos \lambda x \int_0^x q(t) dt - \int_0^x q(t) \cos \lambda (x - 2t) dt + R_2(x, \lambda, q), \quad (1.59)$$

$$c(x, \lambda) = \sin \lambda (\pi - x) \int_x^\pi q(t) dt + \int_x^\pi q(t) \sin \lambda (2t - \pi - x) dt + R_3(x, \lambda, q), \quad (1.60)$$

$$d(x, \lambda) = \cos \lambda (\pi - x) \int_x^\pi q(t) dt + \int_x^\pi q(t) \cos \lambda (\pi + x - 2t) dt + R_4(x, \lambda, q), \quad (1.61)$$

and $R_i, i = 1, 2, 3, 4$, satisfy the estimates (when $|\lambda| \geq 1$)

$$R_1(x, \lambda, q), R_2(x, \lambda, q) = O\left(\frac{e^{|\operatorname{Im}\lambda|x}}{|\lambda|}\right), \quad (1.62)$$

$$R_3(x, \lambda, q), R_4(x, \lambda, q) = O\left(\frac{e^{|\operatorname{Im}\lambda|(\pi-x)}}{|\lambda|}\right), \quad (1.63)$$

uniformly with respect to $q \in BL_C^1[0, \pi]$.

According to (1.53), the solution $\varphi(x, \mu, \alpha, q)$, which we will denote by $\varphi(x, \lambda^2, \alpha)$ for brevity, has the form

$$\varphi(x, \lambda^2, \alpha) = y_1(x, \lambda) \sin \alpha - y_2(x, \lambda) \cos \alpha, \quad (1.64)$$

and according to (1.54) and (1.55) we arrive at:

$$\varphi(x, \lambda^2, \alpha) = \left[\cos \lambda x + \frac{1}{2\lambda} a(x, \lambda) \right] \sin \alpha - \left[\frac{\sin \lambda x}{\lambda} - \frac{1}{2\lambda^2} b(x, \lambda) \right] \cos \alpha.$$

Taking the squares of both sides of the last equality, we obtain:

$$\begin{aligned} \varphi^2(x, \lambda^2, \alpha) &= \cos^2 \lambda x \sin^2 \alpha + \frac{1}{\lambda} \left[a(x, \lambda) \cos \lambda x + \frac{a^2(x, \lambda)}{4\lambda} \right] \sin^2 \alpha - \\ &- \frac{2}{\lambda} \left[\cos \lambda x \sin \lambda x - \frac{b(x, \lambda) \cos \lambda x}{2\lambda} + \frac{a(x, \lambda) \sin \lambda x}{2\lambda} - \frac{a(x, \lambda) b(x, \lambda)}{4\lambda^2} \right] \times \\ &\times \sin \alpha \cos \alpha + \frac{\sin^2 \lambda x}{\lambda^2} \cos^2 \alpha + \left[\frac{b^2(x, \lambda)}{4\lambda^4} - \frac{b(x, \lambda) \sin \lambda x}{\lambda^3} \right] \cos^2 \alpha. \end{aligned} \quad (1.65)$$

Recalling the formulae $\cos^2 \lambda x = \frac{1}{2}(1 + \cos 2\lambda x)$ and $\sin^2 \lambda x = \frac{1}{2}(1 - \cos 2\lambda x)$, from (1.65),

we obtain:

$$\begin{aligned}
\int_0^\pi \varphi^2(x, \lambda^2, \alpha) dx &= \frac{\pi}{2} \sin^2 \alpha + \frac{\sin 2\lambda\pi}{4\lambda} \sin^2 \alpha + \\
&+ \frac{1}{\lambda} \left(\int_0^\pi a(x, \lambda) \cos \lambda x dx + \frac{1}{4\lambda} \int_0^\pi a^2(x, \lambda) dx \right) \sin^2 \alpha - \frac{\sin^2 \lambda\pi}{\lambda^2} \sin \alpha \cos \alpha + \\
&+ \frac{1}{\lambda^2} \left(\int_0^\pi b(x, \lambda) \cos \lambda x dx - \int_0^\pi a(x, \lambda) \sin \lambda x dx \right) \sin \alpha \cos \alpha + \\
&+ \frac{\sin \alpha \cos \alpha}{2\lambda^3} \int_0^\pi a(x, \lambda) b(x, \lambda) dx + \frac{\pi}{2\lambda^2} \cos^2 \alpha - \frac{\sin 2\lambda\pi}{4\lambda^3} \cos^2 \alpha - \\
&\quad - \frac{1}{\lambda^3} \left(\int_0^\pi b(x, \lambda) \sin \lambda x dx - \frac{1}{4\lambda} \int_0^\pi b^2(x, \lambda) dx \right) \cos^2 \alpha. \quad (1.66)
\end{aligned}$$

We are going to receive the asymptotic formula (1.49) by the substitution $\lambda = \lambda_n(q, \alpha, \beta) = \sqrt{\mu_n(q, \alpha, \beta)}$ in (1.66). To this aim, we estimate each term of the right-hand side of (1.66) for $\lambda = \lambda_n$. It follows from (1.28), (0.17) and (0.18) that

$$\sin 2\pi\lambda_n(q, \alpha, \beta) = O\left(\frac{1}{n}\right), \quad (\alpha, \beta) \in (0, \pi] \times [0, \pi) \quad (1.67)$$

and

$$\cos 2\pi\lambda_n(q, \alpha, \beta) = \begin{cases} 1 - O\left(\frac{1}{n^2}\right), & \alpha, \beta \in (0, \pi) \text{ and } \alpha = \pi, \beta = 0, \\ -1 + O\left(\frac{1}{n^2}\right), & \alpha = \pi, \beta \in (0, \pi) \text{ and } \alpha \in (0, \pi), \beta = 0. \end{cases} \quad (1.68)$$

Thus, the second term

$$\frac{\sin 2\pi\lambda_n}{\lambda_n} \sin^2 \alpha = O\left(\frac{1}{n^2}\right) \sin^2 \alpha. \quad (1.69)$$

Important is the third term: $\frac{1}{\lambda_n} \int_0^\pi a(x, \lambda_n) \cos \lambda_n x dx$. According to (1.58) and (1.62) we have

$$a(x, \lambda_n) = A(x, \lambda_n) + O\left(\frac{1}{\lambda_n}\right), \quad (1.70)$$

where

$$A(x, \lambda_n) = \int_0^x q(t) dt \sin \lambda_n x + \int_0^x q(t) \sin \lambda_n(x - 2t) dt. \quad (1.71)$$

After multiplying both sides by $\cos \lambda_n x$, integrating over $[0, \pi]$ and changing the order of integration we get

$$\begin{aligned} \int_0^\pi A(x, \lambda_n) \cos \lambda_n x dx &= \frac{\sin^2 \lambda_n \pi}{\lambda_n} \int_0^\pi q(t) \cos^2 \lambda_n t dt - \\ &- \frac{\sin 2\lambda_n \pi}{4\lambda_n} \int_0^\pi q(t) \sin 2\lambda_n t dt - \frac{1}{2} \int_0^\pi (\pi - t) q(t) \sin 2\lambda_n t dt. \end{aligned} \quad (1.72)$$

Taking into account the formulae (1.67) and (1.68) and denoting (see (1.50))

$$\tilde{s}_n \equiv \tilde{s}_n(q, \alpha, \beta) := -\frac{1}{2} \int_0^\pi (\pi - t) q(t) \sin 2\lambda_n t dt,$$

we can rewrite (1.72) in the form

$$\int_0^\pi A(x, \lambda_n) \cos \lambda_n x dx = \tilde{s}_n + O\left(\frac{1}{n}\right). \quad (1.73)$$

Since $\sin 2\lambda_n t = \sin 2\left(n + \delta_n + O\left(\frac{1}{n}\right)\right)t = \sin 2(n + \delta_n)t + O\left(\frac{1}{n}\right)$ holds uniformly with respect to $t \in [0, \pi]$, then $\tilde{s}_n = s_n + O\left(\frac{1}{n}\right)$, and therefore the third term of (1.66) has the form

$$\frac{1}{\lambda_n} \int_0^\pi a(x, \lambda_n) \cos \lambda_n x dx = \frac{s_n}{n + \delta_n(\alpha, \beta)} + O\left(\frac{1}{n^2}\right).$$

Now, let us focus on the remained terms of the equality (1.66) for $\lambda = \lambda_n$. The terms from the fourth to the eighth have the coefficient $\frac{1}{\lambda_n^\gamma}$, where $\gamma \geq 2$, and therefore they have the order $O\left(\frac{1}{n^2}\right)$. Concerning the last four terms of (1.66), we observe that both $\frac{\sin 2\pi\lambda_n}{4\lambda_n^3} \cos^2 \alpha$ and $\frac{1}{\lambda_n^4} \int_0^\pi b^2(x, \lambda_n) dx \cos^2 \alpha$ have the same order $O\left(\frac{1}{\lambda_n^4}\right) \cos^2 \alpha$. An important term is $\frac{1}{\lambda_n^3} \int_0^\pi b(x, \lambda_n) \sin \lambda_n x dx$. According to (1.59) and (1.62) we can write $b(x, \lambda_n)$ in the form

$$b(x, \lambda_n) = B(x, \lambda_n) + O\left(\frac{1}{\lambda_n}\right), \quad (1.74)$$

where

$$B(x, \lambda_n) = \int_0^x q(t) dt \cos \lambda_n x - \int_0^x q(t) \cos \lambda_n(x - 2t) dt. \quad (1.75)$$

A simple computation yields:

$$B(x, \lambda_n) \sin \lambda_n x = \int_0^x q(t) dt \sin 2\lambda_n x - \int_0^x q(t) \sin 2\lambda_n t dt - A(x, \lambda_n) \cos \lambda_n x.$$

After integrating the latter equality from 0 to π , changing the order of integration and taking into consideration (1.72) we get:

$$\begin{aligned} \int_0^\pi B(x, \lambda_n) \sin \lambda_n x dx &= -\frac{\cos^2 \lambda_n \pi}{\lambda_n} \int_0^\pi q(t) \sin^2 \lambda_n t dt + \\ &+ \frac{\sin 2\lambda_n \pi}{4\lambda_n} \int_0^\pi q(t) \sin 2\lambda_n t dt - \frac{1}{2} \int_0^\pi (\pi - t) q(t) \sin 2\lambda_n t dt = \\ &= O\left(\frac{1}{\lambda_n}\right) + O\left(\frac{1}{\lambda_n^2}\right) + \tilde{s}_n = s_n + O\left(\frac{1}{n}\right), \end{aligned}$$

and therefore the eleventh term of the equality (1.66) for $\lambda = \lambda_n$ has the form

$$\frac{1}{\lambda_n^3} \int_0^\pi b(x, \lambda_n) \sin \lambda_n x dx = \frac{1}{\lambda_n^2} \left(\frac{s_n}{n + \delta_n(\alpha, \beta)} + O\left(\frac{1}{n^2}\right) \right).$$

Let us remark that from (1.28) we have $\frac{1}{\lambda_n} - \frac{1}{n + \delta_n} = O\left(\frac{1}{n^3}\right)$. Thus,

$$\begin{aligned} a_n(q, \alpha, \beta) &= \frac{\pi}{2} \left[1 + \frac{2s_n}{\pi [n + \delta_n(\alpha, \beta)]} + O\left(\frac{1}{n^2}\right) \right] \sin^2 \alpha + O\left(\frac{1}{n^2}\right) \sin \alpha \cos \alpha + \\ &+ \frac{\pi}{2 [n + \delta_n(\alpha, \beta)]^2} \left[1 + \frac{2s_n}{\pi [n + \delta_n(\alpha, \beta)]} + O\left(\frac{1}{n^2}\right) \right] \cos^2 \alpha. \end{aligned} \quad (1.76)$$

If $\sin \alpha \neq 0$, then $O\left(\frac{1}{n^2}\right) \sin \alpha \cos \alpha$ can be included into the term $O\left(\frac{1}{n^2}\right) \sin^2 \alpha$, and if $\sin \alpha = 0$, then these terms are absent. Finally, we can write (1.76) in the form (1.49). For b_n everything can be done similarly. Theorem 1.8 is proved. \square

Now we establish assertion for the remainders $s_n(q, \alpha, \beta)$ similar to the Theorem 1.6; part (b) and Theorem 1.7. More precisely, we prove the following theorem:

Theorem 1.9 *The function s , defined by the formula*

$$s(x) = \sum_{n=2}^{\infty} \frac{s_n(q, \alpha, \beta)}{n + \delta_n(\alpha, \beta)} \cos(n + \delta_n(\alpha, \beta)) x \quad (1.77)$$

(a) *is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, for both $\alpha, \beta \in (0, \pi)$ and $\alpha = \pi, \beta = 0$;*

(b) *is absolutely continuous on $[0, 2\pi]$, for $\alpha = \pi, \beta \in (0, \pi)$.*

Proof. Denote

$$\sigma(x) := \int_0^x (\pi - t) q(t) dt. \quad (1.78)$$

Now, from (1.50), we have

$$\begin{aligned} \frac{s_n(q, \alpha, \beta)}{n + \delta_n(\alpha, \beta)} &= -\frac{1}{2(n + \delta_n(\alpha, \beta))} \int_0^\pi (\pi - t) q(t) \sin 2(n + \delta_n(\alpha, \beta)) t dt = \\ &= -\frac{1}{2(n + \delta_n(\alpha, \beta))} \int_0^\pi \sin 2(n + \delta_n(\alpha, \beta)) t d\sigma(t) = -\frac{\sigma(\pi) \sin 2\pi\delta_n(\alpha, \beta)}{2(n + \delta_n(\alpha, \beta))} + \\ &\quad + \int_0^\pi \sigma(t) \cos 2(n + \delta_n(\alpha, \beta)) t dt. \end{aligned} \quad (1.79)$$

It was observed in (1.67) that $\sin 2\pi\delta_n(\alpha, \beta) = O\left(\frac{1}{n}\right)$. If we denote by $\tilde{\sigma}(x) := \sigma\left(\frac{x}{2}\right)$ and $c_n := \frac{\sin 2\pi\delta_n(\alpha, \beta)}{2(n + \delta_n(\alpha, \beta))} = O\left(\frac{1}{n^2}\right)$, then we can rewrite $s(x)$ in the form

$$s(x) = s_1(x) + s_2(x), \quad (1.80)$$

where

$$s_1(x) = -\sigma(\pi) \sum_{n=2}^{\infty} c_n \cos[n + \delta_n(\alpha, \beta)]x, \quad (1.81)$$

$$s_2(x) = \sum_{n=2}^{\infty} \int_0^{2\pi} \tilde{\sigma}(t) \cos[n + \delta_n(\alpha, \beta)] t dt \cos[n + \delta_n(\alpha, \beta)]x. \quad (1.82)$$

Since $c_n = O\left(\frac{1}{n^2}\right)$, then the series in (1.81) converges absolutely and uniformly on $[0, 2\pi]$, and $s_1 \in AC[0, 2\pi]$.

Next, we consider three cases:

Case I: If $\alpha, \beta \in (0, \pi)$, then by (0.17) and (0.18) we have

$$\delta_n(\alpha, \beta) = \frac{\cot \beta - \cot \alpha}{\pi n} + O\left(\frac{1}{n^2}\right) = \frac{d}{n} + O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n}\right),$$

where $d = \frac{\cot \beta - \cot \alpha}{\pi}$. Recalling the Maclaurin expansions of the functions $\sin x$ and $\cos x$ around the point $x = 0$, we obtain

$$\cos[n + \delta_n(\alpha, \beta)]x = \cos nx - d \cdot x \frac{\sin nx}{n} + e_n(x), \quad (1.83)$$

where $e_n(x)$, as all the other entries of (1.83), is a smooth function ($e_n \in C^\infty$) and

$$e_n(x) = O\left(\frac{1}{n^2}\right) \quad (1.84)$$

uniformly on $x \in [0, 2\pi]$. Therefore k_2 can be written in the form

$$s_2(x) = \tau_1(x) + \tau_2(x) + \tau_3(x),$$

where

$$\begin{aligned} \tau_1(x) = & -d \cdot x \sum_{n=2}^{\infty} \frac{1}{n} \int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt \sin nx - \\ & - d \cdot \sum_{n=2}^{\infty} \frac{1}{n} \int_0^{2\pi} t\tilde{\sigma}(t) \sin ntdt \cos nx + d^2 \cdot x \sum_{n=2}^{\infty} \frac{1}{n^2} \int_0^{2\pi} t\tilde{\sigma}(t) \sin ntdt \sin nx + \\ & + \sum_{n=2}^{\infty} \int_0^{2\pi} e_n(t) \tilde{\sigma}(t) dt \cos nx - d \cdot x \sum_{n=2}^{\infty} \frac{1}{n} \int_0^{2\pi} e_n(t) \tilde{\sigma}(t) dt \sin nx, \end{aligned} \quad (1.85)$$

$$\begin{aligned} \tau_2(x) = & \sum_{n=2}^{\infty} e_n(x) \int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt - \\ & - d \cdot \sum_{n=2}^{\infty} \frac{e_n(x)}{n} \int_0^{2\pi} t\tilde{\sigma}(t) \sin ntdt + \sum_{n=2}^{\infty} e_n(x) \int_0^{2\pi} e_n(t) \tilde{\sigma}(t) \sin ntdt, \end{aligned} \quad (1.86)$$

$$\tau_3(x) = \sum_{n=2}^{\infty} \int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt \cos nx. \quad (1.87)$$

Since $\tilde{\sigma} \in AC[0, 2\pi]$, then Fourier coefficients are

$$\int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt = O\left(\frac{1}{n}\right), \quad \int_0^{2\pi} t\tilde{\sigma}(t) \sin ntdt = O\left(\frac{1}{n}\right). \quad (1.88)$$

Also we note that

$$\int_0^{2\pi} e_n(t) \tilde{\sigma}(t) dt = O\left(\frac{1}{n^2}\right). \quad (1.89)$$

Therefore the trigonometric series in (1.85) converges absolutely and uniformly on $[0, 2\pi]$, and

$\tau_1 \in AC(0, 2\pi)$.

It follows from (1.84), (1.88) and (1.89) that the terms of the series in (1.86) have the order $O\left(\frac{1}{n^3}\right)$, and therefore $\tau_2 \in AC[0, 2\pi]$.

About $l_3(x)$ we can say the following: Since $\tilde{\sigma} \in AC[0, 2\pi]$, then the Fourier series of $\tilde{\sigma}$

$$\tilde{\sigma}(x) = \frac{a_0(\tilde{\sigma})}{2} + \sum_{n=1}^{\infty} (a_n(\tilde{\sigma}) \cos nx + b_n(\tilde{\sigma}) \sin x),$$

where $a_n(\tilde{\sigma}) = \frac{1}{\pi} \int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt$, $b_n(\tilde{\sigma}) = \frac{1}{\pi} \int_0^{2\pi} \tilde{\sigma}(t) \sin ntdt$, converges to $\tilde{\sigma}(x)$ in every point of $[0, 2\pi]$ and this series is a function from $AC[0, 2\pi]$. The same is true for $\sigma^*(x) = \tilde{\sigma}(2\pi - x)$:

$$\tilde{\sigma}(2\pi - x) = \frac{a_0(\sigma^*)}{2} + \sum_{n=1}^{\infty} (a_n(\sigma^*) \cos nx + b_n(\sigma^*) \sin x).$$

But it is easy to see, that $a_n(\sigma^*) = a_n(\tilde{\sigma})$ and $b_n(\sigma^*) = -b_n(\tilde{\sigma})$. So

$$\frac{1}{2}(\tilde{\sigma}(x) + \tilde{\sigma}(2\pi - x)) = \frac{a_0(\tilde{\sigma})}{2} + \sum_{n=1}^{\infty} a_n(\tilde{\sigma}) \cos nx,$$

i.e. this is “the even part” of Fourier series of $\tilde{\sigma}(x)$, and is absolutely continuous on $[0, 2\pi]$.

Thus, part (a) of the theorem 1.9 for the case $\alpha, \beta \in (0, \pi)$ is proved.

Case II: If $\alpha = \pi, \beta = 0$, then $\delta_n(\pi, 0) = 1$, and the function $s_2(\cdot)$ takes the form $s_2(x) = \sum_{n=3}^{\infty} \int_0^{2\pi} \tilde{\sigma}(t) \cos ntdt \cos nx$ and again it is “the even part” of Fourier series (without the zeroth, the first and the second terms) of an absolutely continuous function. Part (a) of the theorem 1.9 is proved.

Case III: Let $\alpha = \pi, \beta \in (0, \pi)$. Since (see (1.33))

$$\begin{aligned} \sin 2(n + \delta_n(\pi, \beta))x - \sin 2\left(n + \frac{1}{2}\right)x &= \\ &= 2 \cos\left(2n + \delta_n(\pi, \beta) + \frac{1}{2}\right)x \sin\left(\delta_n(\pi, \beta) - \frac{1}{2}\right)x = O\left(\frac{1}{n}\right), \end{aligned}$$

$$\begin{aligned} \cos(n + \delta_n(\pi, \beta))x - \cos\left(n + \frac{1}{2}\right)x &= \\ &= -2 \sin\left(n + \frac{2\delta_n(\pi, \beta) + 1}{4}\right)x \sin\left(\frac{2\delta_n(\pi, \beta) - 1}{4}\right)x = O\left(\frac{1}{n}\right) \end{aligned}$$

uniformly on $[0, 2\pi]$, then $s \in AC[0, 2\pi]$ is equivalent to $\tilde{s} \in AC[0, 2\pi]$, where \tilde{s} defined by the formula

$$\tilde{s}(x) := \sum_{n=0}^{\infty} \frac{\tilde{s}_n(q, \pi, \beta)}{n + \frac{1}{2}} \cos\left(n + \frac{1}{2}\right)x, \quad (1.90)$$

where

$$\tilde{s}_n(q, \pi, \beta) = -\frac{1}{2} \int_0^{\pi} (\pi - t) q(t) \sin 2\left(n + \frac{1}{2}\right)tdt.$$

Denote $\sigma_3(x) = \int_0^x (\pi - t) q(t) dt$ and write $\frac{\tilde{s}_n(q, \pi, \beta)}{n + \frac{1}{2}}$ in the following form:

$$\begin{aligned} \frac{\tilde{s}_n(q, \pi, \beta)}{n + \frac{1}{2}} &= -\frac{1}{2(n + \frac{1}{2})} \int_0^\pi (\pi - t) q(t) \sin 2\left(n + \frac{1}{2}\right) t dt = \\ &= -\frac{1}{2(n + \frac{1}{2})} \int_0^\pi \sin 2\left(n + \frac{1}{2}\right) t d\sigma_3(t) = \int_0^\pi \sigma_3(t) \cos 2\left(n + \frac{1}{2}\right) t dt = \\ &= \frac{1}{2} \int_0^{2\pi} \sigma_4(t) \cos\left(n + \frac{1}{2}\right) t dt, \quad (1.91) \end{aligned}$$

where $\sigma_4(x) \equiv \sigma_3\left(\frac{x}{2}\right)$ is the absolutely continuous function on $[0, 2\pi]$.

Since

$$\begin{aligned} \int_\pi^{2\pi} \sigma_4(t) \cos\left(n + \frac{1}{2}\right) t dt &= \int_{-2\pi}^{-\pi} \sigma_4(-t) \cos\left(n + \frac{1}{2}\right) t dt = \\ &= \int_0^\pi \sigma_4(2\pi - t) \cos\left(n + \frac{1}{2}\right) (t - 2\pi) dt = -\int_0^\pi \sigma_4(2\pi - t) \cos\left(n + \frac{1}{2}\right) t dt, \end{aligned}$$

then the integral in (1.91) can be written in the following form

$$\frac{1}{2} \int_0^{2\pi} \sigma_4(t) \cos\left(n + \frac{1}{2}\right) t dt = \frac{1}{2} \int_0^\pi (\sigma_4(t) - \sigma_4(2\pi - t)) \cos\left(n + \frac{1}{2}\right) t dt. \quad (1.92)$$

Taking into account that the system of functions $\left\{ \cos\left(n + \frac{1}{2}\right) x \right\}_{n=0}^\infty$ are the eigenfunctions of the problem $L\left(0, \frac{\pi}{2}, 0\right)$ and using Theorem 1.4 and Remark 1.2, we receive

$$\begin{aligned} \sigma_4(x) - \sigma_4(2\pi - x) &= \\ &= \frac{2}{\pi} \sum_{n=0}^\infty \int_0^\pi (\sigma_4(t) - \sigma_4(2\pi - t)) \cos\left(n + \frac{1}{2}\right) t dt \cos\left(n + \frac{1}{2}\right) x, \quad (1.93) \end{aligned}$$

where the series converges uniformly on $[0, \pi]$.

From (1.90), (1.91), (1.92) and (1.93) we get that

$$\tilde{s}(x) = \frac{\pi}{4} (\sigma_4(x) - \sigma_4(2\pi - x)), \quad x \in [0, \pi],$$

but on the other hand (see (1.90))

$$\tilde{s}(2\pi - x) = -\tilde{s}(x)$$

and therefore \tilde{s} is an absolutely continuous function on $[0, 2\pi]$. This completes the proof. \square

Remark 1.4 *We have proved that in the direct problem (q and β are given) the properties (0.19), (0.21) of the eigenvalues and the properties (0.20), (0.22) of the norming constants take place. This means that the necessary part of the Theorem 0.4 is proved.*

At the end of this section we formulate two theorems on more precise asymptotic formulae for eigenvalues and norming constants, when the potential q is an absolutely continuous function.

Theorem 1.10 *Let $q \in AC_{\mathbb{R}}[0, \pi]$. Then the asymptotic relation ($n \rightarrow \infty$)*

$$\sqrt{\mu_n(q, \pi, \beta)} \equiv \lambda_n(q, \pi, \beta) = n + \delta_n(\pi, \beta) + \frac{[q]}{2(n + \delta_n(\pi, \beta))} + l_n(q, \pi, \beta) + O\left(\frac{1}{n^3}\right), \quad (1.94)$$

holds, where $[q] = \frac{1}{\pi} \int_0^{\pi} q(t) dt$,

$$l_n = l_n(q, \pi, \beta) = \frac{1}{4\pi(n + \delta_n(\pi, \beta))^2} \int_0^{\pi} q'(x) \sin 2(n + \delta_n(\pi, \beta)) x dx. \quad (1.95)$$

The estimate $O\left(\frac{1}{n^3}\right)$ of the remainder in (1.94) is uniform in all $\beta \in [0, \pi]$ and $q' \in BL_{\mathbb{R}}^1[0, \pi]$, and the function (the derivative of the function defined by (1.31))

$$l'(x, \beta) = \sum_{n=2}^{\infty} l_n(q, \pi, \beta) (n + \delta_n(\pi, \beta)) \cos(n + \delta_n(\pi, \beta)) x \quad (1.96)$$

is absolutely continuous on $[0, 2\pi]$, that is $l' \in AC[0, 2\pi]$.

Theorem 1.11 *Let $q \in AC_{\mathbb{R}}[0, \pi]$. Then the asymptotic relation ($n \rightarrow \infty$) for norming constants*

$$a_n(q, \pi, \beta) = \frac{\pi}{2(n + \delta_n(\pi, \beta))^2} \left(1 + \frac{[q]_{\beta}}{2(n + \delta_n(\pi, \beta))^2} + \frac{2s_n}{\pi(n + \delta_n(\pi, \beta))^2} + O\left(\frac{1}{n^3}\right) \right), \quad (1.97)$$

holds, where $[q]_{\beta} = \frac{5}{\pi} \int_0^{\pi} q(t) dt + 2(q(0) + \cot \beta)$,

$$s_n = s_n(q, \pi, \beta) = \frac{1}{4} \int_0^{\pi} (\pi - t) q'(t) \cos 2(n + \delta_n(\pi, \beta)) t dt. \quad (1.98)$$

The estimate $O\left(\frac{1}{n^3}\right)$ of the remainder in (1.97) is uniform in all $\beta \in [0, \pi]$ and $q' \in BL_{\mathbb{R}}^1[0, \pi]$, and the function (the derivative of the function defined by (1.77))

$$s'(x, \beta) = - \sum_{n=2}^{\infty} \frac{s_n(q, \pi, \beta)}{n + \delta_n(\pi, \beta)} \sin(n + \delta_n(\pi, \beta))x \quad (1.99)$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, that is $s' \in AC(0, 2\pi)$.

The proof of these theorems is based on the fact that for the case $q \in AC_{\mathbb{R}}[0, \pi]$ more precise asymptotic formula ($n \rightarrow \infty$) for the eigenfunctions

$$\begin{aligned} \varphi(x, \mu_n(q, \pi, \beta), \pi) &= \frac{\sin \lambda_n x}{\lambda_n} - \frac{\cos \lambda_n x}{2\lambda_n^2} \int_0^x q(t) dt + \\ &+ \frac{\sin \lambda_n x}{4\lambda_n^3} \left(q(x) + q(0) - \frac{1}{2} \left(\int_0^x q(t) dt \right)^2 \right) + \frac{1}{4\lambda_n^3} \int_0^x q'(t) \sin \lambda_n(x - 2t) dt + O\left(\frac{1}{n^4}\right). \end{aligned}$$

holds (can be obtained using integration by parts method). The rest part of the proof is based on the same technique as for general case $q \in L_{\mathbb{R}}^1[0, \pi]$ (see [72, the proof of Theorem 1.6] and the proofs of the Theorems 1.8, 1.9), that is why we omit it.

1.4 Riesz Bases Generated by the Spectra of Sturm-Liouville Problems

It is well-known that the system of eigenfunctions of the problem $L(q, \alpha, \beta)$ forms an orthogonal basis in $L^2[0, \pi]$. In the simplest case, when $q(x) = 0$ almost everywhere (a.e.) on $[0, \pi]$, eigenfunctions of the problem $L(0, \alpha, \beta)$, which satisfy the initial conditions (0.7), have the following form:

$$\varphi_n^0(x, \alpha, \beta) = \cos \lambda_n(0, \alpha, \beta) x \sin \alpha - \frac{\sin \lambda_n(0, \alpha, \beta) x}{\lambda_n(0, \alpha, \beta)} \cos \alpha, \quad n = 0, 1, 2, \dots \quad (1.100)$$

Here rises a natural question: *Do the systems of functions $\{\cos \lambda_n(0, \alpha, \beta) x\}_{n=0}^{\infty}$ and $\{\sin \lambda_n(0, \alpha, \beta) x\}_{n=0}^{\infty}$ separately form basis in $L^2[0, \pi]$?*

Examples show, that the answer is not always positive and depends on α and β . When $\alpha = \beta = \frac{\pi}{2}$, then $\lambda_n\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) = n$, $n = 0, 1, 2, \dots$ and the system $\left\{\cos \lambda_n\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) x\right\}_{n=0}^{\infty} = \{\cos nx\}_{n=0}^{\infty}$ forms an orthogonal basis, but the system $\left\{\sin \lambda_n\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) x\right\}_{n=0}^{\infty} = \{0\} \cup$

$\{\sin nx\}_{n=1}^{\infty}$ is not a basis because of the “unnecessary” member $\sin(0x) \equiv 0$. However, throwing away this “unnecessary” member, we obtain an orthogonal basis $\{\sin nx\}_{n=1}^{\infty}$. In the case of $\alpha = \pi$, $\beta = 0$, $\lambda_n(0, \pi, 0) = n + 1$, $n = 0, 1, 2, \dots$ and the system $\{\sin \lambda_n(0, \pi, 0)x\}_{n=0}^{\infty} = \{\sin(n+1)x\}_{n=0}^{\infty}$ forms an orthogonal basis, but the system $\{\cos \lambda_n(0, \pi, 0)x\}_{n=0}^{\infty} = \{\cos(n+1)x\}_{n=0}^{\infty}$ is not complete in $L^2[0, \pi]$, there is a lack of constant, but adding it, thus, taking the system $\{1\} \cup \{\cos(n+1)x\}_{n=0}^{\infty} = \{\cos nx\}_{n=0}^{\infty}$ we obtain a basis in $L^2[0, \pi]$. The question that we want to answer is the following: Do the systems $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ and $\{\sin \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ form Riesz bases in $L^2[0, \pi]$? The answer we formulate in Theorems 1.14 and 1.15, but before that we recall some basic concepts and results (for equivalent definitions and statements see [78–80]).

Definition 1.1 ([62]) *A basis $\{f_j\}_{j=1}^{\infty}$ of a separable Hilbert space H is called a Riesz basis if it is derived from an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ by linear bounded invertible operator A , i.e., if $f_j = Ae_j$, $j = 1, 2, \dots$*

Definition 1.2 ([62]) *Two sequences of elements $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ from H are called quadratically close if $\sum_{j=1}^{\infty} \|f_j - g_j\|^2 < \infty$.*

Definition 1.3 ([62]) *A sequence $\{g_n\}_{n=0}^{\infty}$ is called ω -linearly independent, if the equality $\sum_{n=0}^{\infty} c_n g_n = 0$ is possible only when $c_n = 0$ for $n = 0, 1, 2, \dots$*

Lemma 1.3 ([62]) *Let $\{f_n\}_{n=0}^{\infty}$ be a Riesz basis in H , $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are quadratically close. If $\{g_n\}_{n=0}^{\infty}$ is ω -linearly independent, then $\{g_n\}_{n=0}^{\infty}$ is a Riesz basis in H .*

Lemma 1.4 ([62]) *Let $\{f_n\}_{n=0}^{\infty}$ be a Riesz basis in H , $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are quadratically close. If $\{g_n\}_{n=0}^{\infty}$ is complete in H , then $\{g_n\}_{n=0}^{\infty}$ is ω -linearly independent (and therefore, is a Riesz basis in H).*

The Riesz basicity of the systems of functions of sines and cosines in $L^2[0, \pi]$ has been studied in many papers (see, for example, [62, 79, 81–84]) and is also associated with Riesz basicity in $L^2[-\pi, \pi]$ the systems of the form $\{e^{i\lambda_n x}\}_{n=-\infty}^{\infty}$ (see, e.g. [80, 85, 86]). Completeness and Riesz basicity of systems of sines and cosines are used in many related areas of mathematics,

in particular, in solutions of the inverse problems in spectral theory of operators (see, e.g. [45, 47, 48, 59, 62]).

The following two theorems have been proved in [79].

Theorem 1.12 ([79]) *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers with the property that $\lambda_k \neq \lambda_m$ for $k \neq m$ and of the form $\lambda_n = n + \delta + \delta_n$, with $\delta_n \in [-l, l]$ for sufficiently large n , where the constants $\delta \in \left[0, \frac{1}{2}\right]$ and $l \in \left(0, \frac{1}{4}\right)$ satisfy $(1 + \sin 2\pi\delta)^{\frac{1}{2}} (1 - \cos \pi l) + \sin \pi l < 1$. Then $\{\cos \lambda_n x\}_{n=0}^{\infty}$ is a Riesz basis in $L^2 [0, \pi]$.*

Theorem 1.13 ([79]) *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers of the form $\lambda_n = n - \delta + \delta_n$, having the same properties as in Theorem 1.12. Then $\{\sin \lambda_n x\}_{n=1}^{\infty}$ is a Riesz basis in $L^2 [0, \pi]$.*

Main results of this section are the following two theorems.

Theorem 1.14 *The system of functions $\{\cos \lambda_n (q, \alpha, \beta) x\}_{n=0}^{\infty}$ is a Riesz basis in $L^2 [0, \pi]$ for each triple $(q, \alpha, \beta) \in L^1_{\mathbb{R}} [0, \pi] \times (0, \pi) \times [0, \pi)$, except one case: when $\alpha = \pi$, $\beta = 0$, the system $\{\cos \lambda_n (q, \pi, 0) x\}_{n=0}^{\infty}$ is not a basis, but the system $\{\cos \lambda x\} \cup \{\cos \lambda_n (q, \pi, 0) x\}_{n=0}^{\infty}$ is a Riesz basis in $L^2 [0, \pi]$, where $\lambda^2 \neq \lambda_n^2$ for every $n = 0, 1, 2, \dots$*

Theorem 1.15 1. *Let $\alpha, \beta \in (0, \pi)$. Then the systems*

(a) $\{\sin \lambda_n x\}_{n=1}^{\infty}$, *if there is no zeros among $\lambda_n = \lambda_n (q, \alpha, \beta)$, $n = 0, 1, 2, \dots$ (i.e. in this case we “throw away” $\sin \lambda_0 x$),*

(b) $\{\sin \lambda_n x\}_{n=0}^{n_0-1} \cup \{\sin \lambda_n x\}_{n=n_0+1}^{\infty}$, *if $\lambda_{n_0} (q, \alpha, \beta) = 0$ (we “throw away” $\sin \lambda_{n_0} x \equiv 0$)*

are Riesz bases in $L^2 [0, \pi]$.

2. *Let $\alpha = \pi$, $\beta \in (0, \pi)$ or $\alpha \in (0, \pi)$, $\beta = 0$. Then the systems*

(a) $\{\sin \lambda_n x\}_{n=0}^{\infty}$, *if there is no zeros among $\lambda_n = \lambda_n (q, \alpha, \beta)$, $n = 0, 1, 2, \dots$,*

(b) $\{\sin \lambda_n x\}_{n=0}^{n_0-1} \cup \{x\} \cup \{\sin \lambda_n x\}_{n=n_0+1}^{\infty}$, *if $\lambda_{n_0} = 0$*

are Riesz bases in $L^2 [0, \pi]$.

3. Let $\alpha = \pi$, $\beta = 0$. The answer is the same as in case 2.

Proof. From (0.17) and (0.18) easily follows that $\delta_n(\alpha, \beta) = O(n^{-1})$ for $\alpha, \beta \in (0, \pi)$; $\delta_n(\alpha, \beta) = \frac{1}{2} + O(n^{-1})$ for $\alpha = \pi$, $\beta \in (0, \pi)$ and $\alpha \in (0, \pi)$, $\beta = 0$; and $\delta_n(\pi, 0) = 1$ for all $n = 2, 3, \dots$. Thus, we distinguish 3 cases:

1. $\alpha, \beta \in (0, \pi)$; i.e. the interior points of the square $[0, \pi] \times [0, \pi]$, where $\lambda_n = \lambda_n(q, \alpha, \beta)$ have the asymptotic property $\lambda_n = n + O(n^{-1})$,
2. $\alpha = \pi$, $\beta \in (0, \pi)$ or $\alpha \in (0, \pi)$, $\beta = 0$ (i.e. right and bottom edges of the square $[0, \pi] \times [0, \pi]$), where λ_n have the asymptotic property $\lambda_n = n + \frac{1}{2} + O(n^{-1})$,
3. $\alpha = \pi$, $\beta = 0$, where $\lambda_n(q, \pi, 0) = n + 1 + O(n^{-1})$.

It follows from (1.28) that the only circumstance, (essentially) preventing us to apply Theorems 1.12 and 1.13 for proving Riesz basicity of systems $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ and $\{\sin \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$, is that among the eigenvalues $\mu_n = \lambda_n^2(q, \alpha, \beta)$ may be negative (see (1.26)), and accordingly among $\lambda_n(q, \alpha, \beta)$ may be (in a finite number) pure imaginary ones. Can these λ_n interfere the Riesz basicity of the mentioned systems? Our answer is contained in Theorems 1.14 and 1.15.

We will start with a lemma, which is an analogue of [79, Lemma 4].

Lemma 1.5 *Let $\{\nu_n^2\}_{n=0}^{\infty}$ and $\{\lambda_n^2\}_{n=0}^{\infty}$ be two real sequences such that $\nu_k^2 \neq \nu_m^2$ and $\lambda_k^2 \neq \lambda_m^2$, for $k \neq m$, and among which only a finite number of members $(\nu_0^2, \nu_1^2, \dots, \nu_{n_1}^2; \lambda_0^2, \lambda_1^2, \dots, \lambda_{n_2}^2)$ can be negative, and the sequences are enumerated in increasing order $(\nu_0^2 < \nu_1^2 < \dots < \nu_n^2 < \dots; \lambda_0^2 < \lambda_1^2 < \dots < \lambda_n^2 < \dots)$. Let $\{\nu_n\}$ and $\{\lambda_n\}$ have the asymptotic properties*

$$\nu_n = n + \delta + O(n^{-1}), \quad 0 \leq \delta \leq 1, \quad (1.101)$$

$$\lambda_n = n + \delta_n(\alpha, \beta) + O(n^{-1}), \quad (1.102)$$

when $n \rightarrow \infty$ and, furthermore,

$$\sum_{n=0}^{\infty} |\lambda_n - \nu_n|^2 < \infty. \quad (1.103)$$

Then $\{\cos \nu_n x\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$ if and only if $\{\cos \lambda_n x\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$.

Proof. Set $f_n(x) = \cos \nu_n x$ and $g_n(x) = \cos \lambda_n x$, $n = 0, 1, 2, \dots$. Assume $\{f_n\}_{n=0}^\infty$ is a Riesz basis in $L^2[0, \pi]$. Since for real numbers ν_n and λ_n ,

$$\begin{aligned} |\cos \nu_n x - \cos \lambda_n x| &= \left| 2 \sin \frac{(\lambda_n - \nu_n)x}{2} \sin \frac{(\lambda_n + \nu_n)x}{2} \right| \leq \\ &\leq 2 \left| \sin \frac{(\lambda_n - \nu_n)x}{2} \right| \leq |\nu_n - \lambda_n| x \leq \pi |\nu_n - \lambda_n|, \end{aligned}$$

we obtain that

$$\|\cos \nu_n x - \cos \lambda_n x\|^2 = \int_0^\pi |\cos \nu_n x - \cos \lambda_n x|^2 dx \leq \pi^3 |\nu_n - \lambda_n|^2.$$

Therefore,

$$\sum_{n=0}^\infty \|f_n - g_n\|^2 = \sum_{n=0}^{n_0} \|f_n - g_n\|^2 + \sum_{n=n_0+1}^\infty \|f_n - g_n\|^2 \leq M_0 + \pi^3 \sum_{n=n_0+1}^\infty |\lambda_n - \nu_n|^2 < \infty,$$

i.e., $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ are quadratically close ($n_0 = \max\{n_1, n_2\}$). According to Lemma 1.3, to prove the Riesz basicity of the system $\{g_n\}_{n=0}^\infty$ it is enough to prove its ω -linearly independence. Assume the contrary, i.e. let there is a sequence $\{c_n\}_{n=0}^\infty \in l^2$, not identically zero, such that

$$\sum_{n=0}^\infty c_n g_n = 0. \quad (1.104)$$

Let $\lambda \in \mathbb{C}$ be such that $\lambda \neq \pm \lambda_n$, $n = 0, 1, 2, \dots$, and define the function

$$g(x) = \sum_{n=0}^\infty \frac{c_n}{\lambda_n^2 - \lambda^2} g_n(x). \quad (1.105)$$

It follows from (1.27) that this series is uniformly convergent for $x \in [0, \pi]$. Similarly, the series

$$g'(x) = - \sum_{n=0}^\infty \frac{c_n \lambda_n}{\lambda_n^2 - \lambda^2} \sin \lambda_n x \quad (1.106)$$

converges uniformly on $[0, \pi]$. Since $g'' = -\lambda^2 g$, we have (note, that here we repeat the proof of [79])

$$\sum_{n=0}^m \frac{c_n}{\lambda_n^2 - \lambda^2} g_n''(x) = - \sum_{n=0}^m \frac{c_n \lambda_n^2}{\lambda_n^2 - \lambda^2} g_n(x) = - \sum_{n=0}^m c_n g_n(x) - \lambda^2 \sum_{n=0}^m \frac{c_n}{\lambda_n^2 - \lambda^2} g_n(x).$$

Taking into account (1.104), we conclude that the sequence on the left-side of the last equality converges in $L^2[0, \pi]$ to $-\lambda^2 g(x)$, when $m \rightarrow \infty$. This implies that g is twice differentiable

and satisfies the differential equation $-g''(x) = \lambda^2 g(x)$, $x \in (0, \pi)$, and initial conditions (see (1.105) and (1.106)):

$$g(0) = h(\lambda) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n^2 - \lambda^2}, \quad g'(0) = 0; \quad (1.107)$$

i.e., g is the solution of the corresponding Cauchy problem, which is unique and given by the formula

$$g(x) = h(\lambda) \cos \lambda x. \quad (1.108)$$

The function $h(\lambda)$ defined by (1.107) is meromorphic, and taking into account that $\{c_n\}_{n=0}^{\infty} \neq \{0\}_{n=0}^{\infty}$, is not an identically zero function. Then it has no more than countable number of isolated zeros. If $h(\lambda) \neq 0$, then (1.105) and (1.108) show that $\cos \lambda x$ belongs to the closed linear span of the system $\{g_n\}_{n=0}^{\infty}$ in $L^2[0, \pi]$. Since $\cos \lambda x$ is a continuous function of (λ, x) , we obtain that $\cos \lambda x$ belongs to closed linear span of the system $\{g_n\}_{n=0}^{\infty}$ for all $\lambda \in \mathbb{C}$. Particularly, the all $\cos nx$, $n = 0, 1, 2, \dots$ belong to the closed linear span of the system $\{g_n\}_{n=0}^{\infty}$, so the system $\{g_n\}_{n=0}^{\infty}$ is a complete system in $L^2[0, \pi]$. From Lemma 1.4 follows the ω -linearly independence of the system $\{g_n\}_{n=0}^{\infty}$; i.e. we come to a contradiction, and the Riesz basicity of the system $\{g_n\}_{n=0}^{\infty}$ is proved. If we assume, that $\{g_n\}_{n=0}^{\infty}$ is a Riesz basis, then similarly we can prove the Riesz basicity of the system $\{f_n\}_{n=0}^{\infty}$. Lemma 1.5 is proved. \square

Let us now turn to the proof of the Theorem 1.14. We start from the first case: $\alpha, \beta \in (0, \pi)$. To this aim, in the Lemma 1.5 we take $\nu_n = n$, $f_n(x) = \cos nx$, and $g_n(x) = \cos \lambda_n(q, \alpha, \beta)x$, $n = 0, 1, 2, \dots$. In this case $\lambda_n(q, \alpha, \beta) = n + O(n^{-1})$, and, therefore, (1.103) holds; i.e., $\{f_n\}$ and $\{g_n\}$ are quadratically close. Since $\{f_n\}_{n=0}^{\infty} = \{\cos nx\}_{n=0}^{\infty}$ is a Riesz basis, then from the Lemma 1.5 follows the Riesz basicity of the system $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$.

In the second case in Lemma 1.5 we take $\nu_n = n + \frac{1}{2}$, $f_n(x) = \cos\left(n + \frac{1}{2}\right)x$ and $g_n(x) = \cos \lambda_n(q, \alpha, \beta)x$, $n = 0, 1, 2, \dots$. In the second case $\lambda_n(q, \alpha, \beta) = n + \frac{1}{2} + O(n^{-1})$ and therefore again holds (1.103); i.e. quadratically closeness. As $\left\{\cos\left(n + \frac{1}{2}\right)x\right\}_{n=0}^{\infty}$ is the system of eigenfunctions of the Sturm-Liouville problem $L\left(0, \frac{\pi}{2}, 0\right)$, it is an orthogonal basis in $L^2[0, \pi]$ (and, particularly, is a Riesz basis). From the Lemma 1.5 follows the Riesz basicity of the system $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ in this case.

In the third case in Lemma 1.5 we take $\nu_n = n + 1$ ($\delta = 1$), $f_n(x) = \cos(n + 1)x$ and $g_n(x) = \cos \lambda_n(q, \pi, 0)x$, $n = 0, 1, 2, \dots$. If we assume that $\{g_n\}_{n=0}^\infty$ is a Riesz basis, then from the asymptotic property $\lambda_n(q, \pi, 0) = n + 1 + O(n^{-1})$ and Lemma 1.5 follows the Riesz basicity of the system $\{f_n\}_{n=0}^\infty = \{\cos(n + 1)x\}_{n=0}^\infty$, which is incorrect, since it is even not complete. Therefore, $\{\cos \lambda_n(q, \pi, 0)x\}_{n=0}^\infty$ does not form a Riesz basis. But adding to this system a function $f(x) = \cos \lambda x$, where $\lambda^2 \neq \lambda_n^2$ for every $n = 0, 1, 2, \dots$ and observing that the system $\{f(x)\} \cup \{\cos \lambda_n(q, \pi, 0)x\}_{n=0}^\infty$ is ω -linearly independent and quadratically close to the system $\{\cos nx\}_{n=0}^\infty$, according to the Lemma 1.5, we get its Riesz basicity. Theorem 1.14 is proved. \square

Lemma 1.6 *Let $\{\nu_n^2\}_{n=0}^\infty$ and $\{\lambda_n^2\}_{n=0}^\infty$ be the same as in Lemma 1.5. Then $\{\sin \nu_n x\}_{n=0}^\infty$ is a Riesz basis in $L^2[0, \pi]$ if and only if $\{\sin \lambda_n x\}_{n=0}^\infty$ is a Riesz basis in $L^2[0, \pi]$.*

Proof. Set $f_n(x) = \sin \nu_n x$, $g_n(x) = \sin \lambda_n x$, $n = 0, 1, 2, \dots$. Quadratic closeness of the systems $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ can be showed in the same way as in Lemma 1.5. Function $g(x)$ (see (1.105)) in this case is the solution of the Cauchy problem $-g'' = \lambda^2 g$, $g(0) = 0$, $g'(0) = h_1(\lambda) = \sum_{n=0}^\infty \frac{c_n \lambda_n}{\lambda_n^2 - \lambda^2}$, and, therefore, has the form $g(x) = h_1(\lambda) \frac{\sin \lambda x}{\lambda}$. From the continuity of $\frac{\sin \lambda x}{\lambda}$ as a function of two variables (λ, x) follows that the equality

$$\frac{\sin \lambda x}{\lambda} = \frac{1}{h_1(\lambda)} \sum_{n=0}^\infty \frac{c_n}{\lambda_n^2 - \lambda^2} \sin \lambda_n x \quad (1.109)$$

holds not only when $h_1(\lambda) \neq 0$, but also for all $\lambda \in \mathbb{C}$. Hence (1.109) takes place for $\lambda = 1, 2, 3, \dots$; i.e., the all elements of the orthogonal basis $\{\sin nx\}_{n=1}^\infty$ are in the closed linear span of the system $\{\sin \lambda_n x\}_{n=0}^\infty$; i.e., the system $\{\sin \lambda_n x\}_{n=0}^\infty$ is complete in $L^2[0, \pi]$. The rest of the proof repeats the same lines as in Lemma 1.5. \square

Now the proof of Theorem 1.15 is: In stated in following cases:

(1.a) we take $\nu_n = n + 1$ and accordingly $\{f_n(x)\}_{n=0}^\infty = \{\sin(n + 1)x\}_{n=0}^\infty = \{\sin nx\}_{n=1}^\infty$ and $\{g_n(x)\}_{n=1}^\infty = \{\sin \lambda_n x\}_{n=1}^\infty$, as stated in Theorem 1.15. Since $\{f_n\}_{n=0}^\infty$ is a Riesz basis (and even an orthogonal basis) and from the asymptotic property $\lambda_n = n + O(n^{-1})$ it follows that $\{f_n\}$ and $\{g_n\}$ are quadratically close, therefore from Lemma 1.6 follows the Riesz basicity of the system $\{\sin \lambda_n x\}_{n=1}^\infty$.

(1.b) also $\{f_n(x)\}_{n=1}^\infty = \{\sin nx\}_{n=1}^\infty$ and the system

$$\{\sin \lambda_n x\}_{n=0}^{n_0-1} \cup \{\sin \lambda_n x\}_{n=n_0+1}^\infty$$

is again quadratically close to $\{f_n\}_{n=1}^\infty$.

(2.a) we take $\nu_n = n + \frac{1}{2}$, accordingly, $f_n(x) = \sin\left(n + \frac{1}{2}\right)x$, and $g_n(x) = \sin \lambda_n x$, $n = 0, 1, \dots$. Since $\left\{\sin\left(n + \frac{1}{2}\right)x\right\}_{n=0}^\infty$ is the system of eigenfunctions of the self-adjoint problem $L(0, \pi, \frac{\pi}{2})$, than it is an orthogonal basis in $L^2[0, \pi]$. The asymptotic property $\lambda_n = n + \frac{1}{2} + O(n^{-1})$ ensures the quadratically closeness of the systems $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$, therefore in this case the Riesz basicity of the system $\{g_n\}_{n=0}^\infty$ is proved.

(2.b) again $f_n(x) = \sin\left(n + \frac{1}{2}\right)x$, $n = 0, 1, \dots$, and $\{g_n\}$ is different from the case (2.a) with only one element g_{n_0} , which has not any effect on quadratically closeness of the systems $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$.

(3) we take $\nu_n = n + 1$ and $f_n(x) = \sin(n + 1)x$, $n = 0, 1, 2, \dots$; i.e., $\{f_n(x)\}_{n=0}^\infty = \{\sin nx\}_{n=1}^\infty$. The rest is followed from the asymptotic property $\lambda_n(q, \pi, 0) = n + 1 + O(n^{-1})$, if we take $g_n(x) = \sin \lambda_n(q, \pi, 0)x$, $n = 0, 1, \dots$.

Therefore, Theorem 1.15 is proved. \square

Remark 1.5 From Lemmas 1.5 and 1.6 it easily follows that $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^\infty$ is a Riesz basis in $L^2[0, \pi]$ if and only if $\{\cos \lambda_n(0, \alpha, \beta)x\}_{n=0}^\infty$ is a Riesz basis in $L^2[0, \pi]$. Similarly for sines. This means that the stability of Riesz basicity is not affected by adding the potential $q(\cdot)$.

Constructive Solution of an Inverse Sturm-Liouville

Problem

In this chapter we prove that the conditions (0.19)–(0.22) are sufficient for the two sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ to be the spectrum and the norming constants respectively, for a problem $L(q, \pi, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and some $\beta \in (0, \pi)$. At the same time we provide an efficient algorithm (an analogue of the Gelfand-Levitan algorithm) for the reconstruction of the problem. To this end in Section 2.1 we derive an analogue of the Gelfand-Levitan equation for our case ($\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0, \pi]$). In Section 2.2 the existence and uniqueness of the solution of this Gelfand-Levitan equation as well as reconstruction of the function q (i.e. the reconstruction of differential equation (0.4)) and parameter $\tilde{\beta}$ (see Remark 2.2) are given. In Section 2.3 we provide an example of the implementation of our algorithm (by taking two certain sequences and using reconstruction technique we find the potential q and the parameter $\tilde{\beta}$). In Section 2.4 we prove three auxiliary lemmas, which are used to obtain some results of this work.

2.1 Derivation of an Analogue of the Gelfand-Levitan Equation

Transformation (transmutation) operators play an important role in the theory of the inverse Sturm-Liouville problems.

The following assertion can be found in [62] (see, also [37, 51]):

Theorem 2.1 ([62]) *For the function $\varphi_\pi(x, \mu)$ the following representation holds*

$$\varphi_\pi(x, \mu) = (\mathbb{I} + \mathbb{P}) \frac{\sin \lambda x}{\lambda} := \frac{\sin \lambda x}{\lambda} + \int_0^x P(x, t) \frac{\sin \lambda t}{\lambda} dt, \quad (2.1)$$

where $P(x, t)$, $0 \leq t \leq x \leq \pi$, is a real continuous function with the same smoothness as $\int_0^x q(t)dt$, and

$$P(x, x) = \frac{1}{2} \int_0^x q(t)dt, \quad P(x, 0) = 0. \quad (2.2)$$

The proof of the Theorem 2.1 was given for the case $q \in L_{\mathbb{R}}^2[0, \pi]$, but it can be easily done for the case $q \in L_{\mathbb{R}}^1[0, \pi]$, without any significant changes.

Further, in this section, our goal is to derive the analogue of Gelfand-Levitan equation for our case.

Lemma 2.1 *Let us given two sequences $\{\mu_n\}_{n \geq 0}$ and $\{a_n\}_{n \geq 0}$, which satisfy the conditions of the Theorem 0.4. Then the function H , defined by the formula*

$$H(t) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} \frac{\cos \lambda_n t - 1}{\lambda_n^2} - \frac{1}{a_n(0, \pi, \beta)} \frac{\cos \lambda_n(0, \pi, \beta)t - 1}{\lambda_n^2(0, \pi, \beta)} \right), \quad (2.3)$$

is absolutely continuous on arbitrary segment $[0, b] \subset [0, 2\pi)$ ($H(\cdot) \in AC[0, 2\pi)$).

Proof. Denote

$$k_n := a_n \lambda_n^2, \quad k_n^0 := a_n(0, \pi, \beta) (n + \delta_n(\pi, \beta))^2, \quad \epsilon_n := \frac{c}{2(n + \delta_n(\pi, \beta))} + l_n.$$

Since $\lambda_n^0 := \lambda_n(0, \pi, \beta) = n + \delta_n(\pi, \beta)$, $n = 2, 3, \dots$, then, we can write down the general term ($n \geq 2$) of the series (2.3) in the following form:

$$\begin{aligned} \frac{1}{a_n} \frac{\cos \lambda_n t - 1}{\lambda_n^2} - \frac{1}{a_n(0, \pi, \beta)} \frac{\cos \lambda_n(0, \pi, \beta)t - 1}{\lambda_n^2(0, \pi, \beta)} &= \frac{\cos \lambda_n t - 1}{k_n} - \frac{\cos \lambda_n^0 t - 1}{k_n^0} = \\ &= \frac{1}{k_n} \cos \lambda_n t - \frac{1}{k_n} \cos \lambda_n^0 t + \frac{1}{k_n} \cos \lambda_n^0 t - \frac{1}{k_n^0} \cos \lambda_n^0 t - \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right) = \\ &= \frac{1}{k_n} (\cos(\lambda_n^0 + \epsilon_n)t - \cos \lambda_n^0 t) + \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right) \cos \lambda_n^0 t - \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right) = \\ &= \frac{1}{k_n} (\cos(n + \delta_n(\pi, \beta) + \epsilon_n)t - \cos(n + \delta_n(\pi, \beta))t) + \\ &\quad + \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right) \cos(n + \delta_n(\pi, \beta))t - \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right). \end{aligned} \quad (2.4)$$

Using (0.19), (0.20) and the condition $l_n = o\left(\frac{1}{n}\right)$ we can write $\frac{1}{k_n}$ in the form:

$$\begin{aligned} \frac{1}{k_n} &= \frac{1}{a_n \lambda_n^2} = \frac{2}{\pi \left(1 + \frac{2s_n}{\pi(n + \delta(\pi, \beta))} \right) \left(1 + \frac{c + r_n}{(n + \delta_n(\pi, \beta))^2} \right)} = \\ &= \frac{2}{\pi} - \frac{4}{\pi^2} \frac{s_n}{n + \delta_n(\pi, \beta)} + h_n, \end{aligned} \quad (2.5)$$

where

$$r_n = 2(n + \delta_n(\pi, \beta))l_n + \left(\frac{c}{2(n + \delta_n(\pi, \beta))} + l_n \right)^2 = o(1)$$

and

$$h_n = \frac{-2 \frac{c + r_n}{(n + \delta_n(\pi, \beta))^2}}{\pi \left(1 + \frac{2s_n}{\pi(n + \delta(\pi, \beta))} \right) \left(1 + \frac{c + r_n}{(n + \delta_n(\pi, \beta))^2} \right)} + \frac{\frac{8s_n^2}{\pi^2(n + \delta_n(\pi, \beta))^2}}{\pi \left(1 + \frac{2s_n}{\pi(n + \delta(\pi, \beta))} \right)} = O\left(\frac{1}{n^2}\right).$$

On the other hand (see (1.42)),

$$\frac{1}{k_n^0} = \frac{1}{a_n(0, \pi, \beta)(n + \delta_n(\pi, \beta))^2} = \frac{2}{\pi} + g_n, \quad g_n = O\left(\frac{1}{n^2}\right).$$

Therefore

$$\frac{1}{k_n} - \frac{1}{k_n^0} = -\frac{4}{\pi^2} \frac{s_n}{n + \delta_n(\pi, \beta)} + h_n - g_n. \quad (2.6)$$

And finally, using elementary trigonometric identities and Maclaurin expansions of cos and sin functions

$$\cos \epsilon_n t = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \epsilon_n^{2k} t^{2k}, \quad \sin \epsilon_n t = \epsilon_n t + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \epsilon_n^{2k+1} t^{2k+1},$$

we get that

$$\begin{aligned} & \cos(n + \delta_n(\pi, \beta) + \epsilon_n)t - \cos(n + \delta_n(\pi, \beta))t = \\ & = -\epsilon_n t \sin(n + \delta_n(\pi, \beta))t + m_n(t) \cos(n + \delta_n(\pi, \beta))t - \\ & \quad - z_n(t) \sin(n + \delta_n(\pi, \beta))t, \end{aligned} \quad (2.7)$$

where $m_n(t) = O\left(\frac{1}{n^2}\right)$ and $z_n(t) = O\left(\frac{1}{n^3}\right)$ uniformly with respect to $t \in [0, 2\pi]$.

Now by taking into account (2.4), (2.5), (2.6) and (2.7) we can expand $H(t)$ in the following form:

$$H(t) = \sum_{n=0}^1 \left(\frac{\cos \lambda_n t - 1}{k_n} - \frac{\cos \lambda_n^0 t - 1}{k_n^0} \right) + \sum_{i=1}^5 H_i(t) + \sum_{n=2}^{\infty} \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right), \quad (2.8)$$

where

$$H_1(t) = -\frac{ct}{\pi} \sum_{n=2}^{\infty} \frac{\sin(n + \delta_n(\pi, \beta))t}{n + \delta_n(\pi, \beta)}, \quad (2.9)$$

$$H_2(t) = -\frac{2t}{\pi} \sum_{n=2}^{\infty} l_n \sin(n + \delta_n(\pi, \beta))t = -\frac{2t}{\pi} l(t), \quad (2.10)$$

$$\begin{aligned}
H_3(t) = & - \sum_{n=2}^{\infty} \left(-\frac{4}{\pi^2} \frac{s_n}{n + \delta_n(\pi, \beta)} + h_n \right) \epsilon_n t \sin(n + \delta_n(\pi, \beta)) t + \\
& + \sum_{n=2}^{\infty} \left(\frac{2}{\pi} - \frac{4}{\pi^2} \frac{s_n}{n + \delta_n(\pi, \beta)} + h_n \right) m_n(t) \cos(n + \delta_n(\pi, \beta)) t - \\
& - \sum_{n=2}^{\infty} \left(\frac{2}{\pi} - \frac{4}{\pi^2} \frac{s_n}{n + \delta_n(\pi, \beta)} + h_n \right) z_n(t) \sin(n + \delta_n(\pi, \beta)) t, \quad (2.11)
\end{aligned}$$

$$H_4(t) = -\frac{4}{\pi^2} \sum_{n=2}^{\infty} \frac{s_n}{n + \delta_n(\pi, \beta)} \cos(n + \delta_n(\pi, \beta)) t = -\frac{4}{\pi^2} s(t), \quad (2.12)$$

$$H_5(t) = \sum_{n=2}^{\infty} (h_n - g_n) \cos(n + \delta_n(\pi, \beta)) t. \quad (2.13)$$

Since $T_\beta(x) := \sum_{n=2}^{\infty} \frac{\sin(n + \delta_n(\pi, \beta)) x}{n + \delta_n(\pi, \beta)}$ is an absolutely continuous functions on arbitrary segment $[a, b] \subset (0, 2\pi)$ (see Lemma 2.6) and $T_\beta(0) = 0$, then $H_1 \in AC[0, 2\pi]$. Similarly, $l(t)$ is an absolutely continuous functions on arbitrary segment $[a, b] \subset (0, 2\pi)$ (see (0.21)) and $l(0) = 0$, then $H_2 \in AC[0, 2\pi]$. The estimates $s_n = o(1)$, $h_n = O\left(\frac{1}{n^2}\right)$, $\epsilon_n = O\left(\frac{1}{n}\right)$, $m_n(t) = O\left(\frac{1}{n^2}\right)$ and $z_n(t) = O\left(\frac{1}{n^3}\right)$ ensure the absolutely and uniformly convergence of the series in (2.11) and absolute continuity of the function H_3 on $[0, 2\pi]$. The condition (0.22) implies absolute continuity of the function H_4 on $[0, 2\pi]$. The estimates $h_n = O\left(\frac{1}{n^2}\right)$ and $g_n = O\left(\frac{1}{n^2}\right)$ ensure the absolutely and uniformly convergence of the series in (2.13) and absolute continuity of the function H_5 on $[0, 2\pi]$. Due to $s(0) = \sum_{n=2}^{\infty} \frac{s_n}{n + \delta_n(\pi, \beta)} < +\infty$ (see (0.22)), we have $\sum_{n=2}^{\infty} \left(\frac{1}{k_n} - \frac{1}{k_n^0} \right) < +\infty$ (see (2.6)). Lemma 2.1 is proved. \square

Consider the function

$$F(x, t) := \frac{1}{2} (H(|x - t|) - H(x + t)), \quad (x, t) \in [0, \pi] \times [0, \pi]. \quad (2.14)$$

It follows from (2.3), that

$$F(x, t) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n(0, \pi, \beta) x}{\lambda_n(0, \pi, \beta)} \frac{\sin \lambda_n(0, \pi, \beta) t}{\lambda_n(0, \pi, \beta)} \right). \quad (2.15)$$

According to Lemma 2.1, $F(x, t)$ is a continuous function in $\{[0, \pi] \times [0, \pi]\} \setminus \{(\pi, \pi)\}$ and $\frac{d}{dx} F(x, x) \in L^1_{\mathbb{R}}(0, \pi)$.

Theorem 2.2 For each fixed $x \in (0, \pi]$, the kernel $P(x, t)$ of the transformation operator (see (2.1)) satisfies the following linear integral equation

$$P(x, t) + F(x, t) + \int_0^x P(x, s) F(s, t) ds = 0, \quad 0 \leq t < x, \quad (2.16)$$

which is also called Gelfand-Levitan equation.

Proof. Since \mathbb{P} defined in (2.1) is a Volterra integral operator with continuous kernel $P(x, t)$, then $\mathbb{I} + \mathbb{P}$ has the inverse operator of the same type (see, for example, [37, 45, 48, 87]), which we denote by $\mathbb{I} + \mathbb{Q}$. Solving the equation (2.1) with respect to $\frac{\sin \lambda x}{\lambda}$ we obtain

$$\frac{\sin \lambda x}{\lambda} = (\mathbb{I} + \mathbb{Q}) \varphi_\pi(x, \lambda^2) := \varphi_\pi(x, \lambda^2) + \int_0^x Q(x, t) \varphi_\pi(t, \lambda^2) dt, \quad (2.17)$$

where $Q(x, t)$, $0 \leq t \leq x \leq \pi$ is a real continuous function with the same smoothness as $P(x, t)$, and

$$Q(x, x) = -\frac{1}{2} \int_0^x q(t) dt, \quad Q(x, 0) = 0. \quad (2.18)$$

By taking $\lambda = \lambda_n$ in (2.1) and multiplying both sides by $\frac{\sin \lambda_n t}{\lambda_n}$, we obtain

$$\begin{aligned} \sum_{n=2}^N \frac{\varphi_\pi(x, \lambda_n^2) \frac{\sin \lambda_n t}{\lambda_n}}{a_n} &= \sum_{n=2}^N \frac{1}{a_n} \left(\frac{\sin \lambda_n x}{\lambda_n} + \int_0^x P(x, s) \frac{\sin \lambda_n s}{\lambda_n} ds \right) \frac{\sin \lambda_n t}{\lambda_n} = \\ &= \sum_{n=2}^N \frac{1}{a_n} \left(\frac{\sin \lambda_n x \sin \lambda_n t}{\lambda_n^2} + \frac{\sin \lambda_n t}{\lambda_n} \int_0^x P(x, s) \frac{\sin \lambda_n s}{\lambda_n} ds \right). \end{aligned} \quad (2.19)$$

On the other hand, if, on the left side of the (2.19), instead of $\frac{\sin \lambda_n t}{\lambda_n}$, substitute the expression (2.17) ($\lambda = \lambda_n$), we get

$$\begin{aligned} \sum_{n=2}^N \frac{\varphi_\pi(x, \lambda_n^2) \frac{\sin \lambda_n t}{\lambda_n}}{a_n} &= \sum_{n=2}^N \frac{1}{a_n} \varphi_\pi(x, \lambda_n^2) \left(\varphi_\pi(t, \lambda_n^2) + \int_0^t Q(t, s) \varphi_\pi(s, \lambda_n^2) ds \right) = \\ &= \sum_{n=2}^N \frac{1}{a_n} \left(\varphi_\pi(x, \lambda_n^2) \varphi_\pi(t, \lambda_n^2) + \varphi_\pi(x, \lambda_n^2) \int_0^t Q(t, s) \varphi_\pi(s, \lambda_n^2) ds \right). \end{aligned} \quad (2.20)$$

Let us denote by $\Phi_N(x, t)$ denote the following expression

$$\Phi_N(x, t) := \sum_{n=0}^N \left(\frac{1}{a_n} \varphi_\pi(x, \lambda_n^2) \varphi_\pi(t, \lambda_n^2) - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 x}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right). \quad (2.21)$$

It is easy to calculate that the following representation holds (compare with [62])

$$\Phi_N(x, t) = I_{N1}(x, t) + I_{N2}(x, t) + I_{N3}(x, t) + I_{N4}(x, t), \quad (2.22)$$

where

$$I_{N1}(x, t) = \sum_{n=0}^N \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 x}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) \quad (2.23)$$

$$I_{N2}(x, t) = \sum_{n=0}^N \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \int_0^x P(x, s) \frac{\sin \lambda_n^0 s}{\lambda_n^0} ds, \quad (2.24)$$

$$I_{N3}(x, t) = \sum_{n=0}^N \int_0^x P(x, s) \left(\frac{1}{a_n} \frac{\sin \lambda_n s}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 s}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) ds, \quad (2.25)$$

$$I_{N4}(x, t) = - \sum_{n=0}^N \frac{\varphi_\pi(x, \lambda_n^2)}{a_n} \int_0^t Q(t, s) \varphi_\pi(s, \lambda_n^2) ds. \quad (2.26)$$

Let f be an absolutely continuous function on $[0, \pi]$, such that $f(0) = 0$. Using triangle inequality and Theorem 1.3 with Remark 1.2 we receive

$$\lim_{N \rightarrow \infty} \max_{x \in [0, \pi]} \int_0^\pi f(t) \Phi_N(x, t) dt = 0. \quad (2.27)$$

Extending $P(x, t) = Q(x, t) = 0$ for $x < t$ and using the standard theorems of the limit process under the integral sign we get that uniformly with respect to $x \in [0, \pi]$

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N1}(x, t) dt &= \lim_{N \rightarrow \infty} \int_0^\pi f(t) \sum_{n=0}^N \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \right. \\ &\quad \left. - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 x}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) dt = \int_0^\pi f(t) \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \right. \\ &\quad \left. - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 x}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) dt = \int_0^\pi f(t) F(x, t) dt, \quad (2.28) \end{aligned}$$

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N2}(x, t) dt &= \lim_{N \rightarrow \infty} \int_0^\pi f(t) \sum_{n=0}^N \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \int_0^x P(x, s) \frac{\sin \lambda_n^0 s}{\lambda_n^0} ds dt = \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{a_n(0, \pi, \beta)} \int_0^\pi \int_0^\pi f(t) \frac{\sin \lambda_n^0 t}{\lambda_n^0} P(x, s) \frac{\sin \lambda_n^0 s}{\lambda_n^0} dt ds = \\ &= \int_0^\pi \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{a_n(0, \pi, \beta)} \int_0^\pi f(t) \frac{\sin \lambda_n^0 t}{\lambda_n^0} dt \frac{\sin \lambda_n^0 s}{\lambda_n^0} P(x, s) ds = \\ &= \int_0^\pi f(s) P(x, s) ds = \int_0^\pi f(t) P(x, t) dt, \quad (2.29) \end{aligned}$$

$$\begin{aligned}
\lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N3}(x, t) dt &= \lim_{N \rightarrow \infty} \int_0^\pi f(t) \sum_{n=0}^N \int_0^x P(x, s) \left(\frac{1}{a_n} \frac{\sin \lambda_n s}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \right. \\
&\quad \left. - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 s}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) ds dt = \int_0^\pi f(t) \int_0^x P(x, s) \times \\
&\quad \times \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\frac{1}{a_n} \frac{\sin \lambda_n s}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{1}{a_n(0, \pi, \beta)} \frac{\sin \lambda_n^0 s}{\lambda_n^0} \frac{\sin \lambda_n^0 t}{\lambda_n^0} \right) ds dt = \\
&= \int_0^\pi f(t) \left(\int_0^x P(x, s) F(s, t) ds \right) dt, \quad (2.30)
\end{aligned}$$

$$\begin{aligned}
\lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N4}(x, t) dt &= - \lim_{N \rightarrow \infty} \int_0^\pi f(t) \sum_{n=0}^N \frac{\varphi_\pi(x, \lambda_n^2)}{a_n} \int_0^t Q(t, s) \varphi_\pi(s, \lambda_n^2) ds dt = \\
&= - \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{\varphi_\pi(x, \lambda_n^2)}{a_n} \int_0^\pi f(t) \int_0^t Q(t, s) \varphi_\pi(s, \lambda_n^2) ds dt = \\
&= - \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{\varphi_\pi(x, \lambda_n^2)}{a_n} \int_0^\pi \left(\int_s^\pi Q(t, s) f(t) dt \right) \varphi_\pi(s, \lambda_n^2) ds = - \int_0^\pi Q(t, x) f(t) dt, \quad (2.31)
\end{aligned}$$

where $Q(t, x) = 0$ for $t < x$. From (2.22) and (2.27)–(2.31), we can conclude that

$$\int_0^\pi f(t) \left(F(x, t) + P(x, t) + \int_0^x P(x, s) F(s, t) ds - Q(t, x) \right) dt = 0. \quad (2.32)$$

Since the system of eigenfunctions $\{\varphi_m(t)\}_{m=0}^\infty$ of the boundary value problem $L(q, \pi, \beta)$ is complete in $L^2(0, \pi)$ and $\varphi_m(0) = 0$, $m = 0, 1, 2, \dots$, then we can take $f(t) = \varphi_m(t)$, $m = 0, 1, 2, \dots$ and obtain that for each fixed $x \in (0, \pi]$

$$F(x, t) + P(x, t) + \int_0^x P(x, s) F(s, t) ds - Q(t, x) = 0. \quad (2.33)$$

For $t < x$, this is equivalent to (2.16). This completes the proof. \square

2.2 The Constructive Solution of the Inverse Problem

Lemma 2.2 For each fixed $x \in (0, \pi]$, equation (2.16) has a unique solution $P(x, \cdot)$ in $L^2[0, x)$.

Proof. Since (2.16) is a Fredholm equation, then it is sufficient (see, for example, [87]) to prove that the homogenous equation

$$p(t) + \int_0^x p(s) F(s, t) ds = 0 \quad (2.34)$$

has only a trivial solution $p(t) = 0$.

Let $p(t) \not\equiv 0$ be a solution of (2.34). Then multiplying (2.34) by $p(t)$ and integrating from 0 to x , we receive

$$\int_0^x p^2(t) dt + \int_0^x \int_0^x F(s, t) p(s) p(t) ds dt = 0$$

or

$$\begin{aligned} \int_0^x p^2(t) dt + \sum_{n=0}^{\infty} \frac{1}{a_n} \int_0^x p(s) \frac{\sin \lambda_n s}{\lambda_n} ds \int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt - \\ - \sum_{n=0}^{\infty} \frac{1}{a_n(0, \pi, \beta)} \int_0^x p(s) \frac{\sin \lambda_n(0, \pi, \beta) s}{\lambda_n(0, \pi, \beta)} ds \int_0^x p(t) \frac{\sin \lambda_n(0, \pi, \beta) t}{\lambda_n(0, \pi, \beta)} dt = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \int_0^x p^2(t) dt + \sum_{n=0}^{\infty} \frac{1}{a_n} \left(\int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt \right)^2 - \\ - \sum_{n=0}^{\infty} \frac{1}{a_n(0, \pi, \beta)} \left(\int_0^x p(t) \frac{\sin \lambda_n(0, \pi, \beta) t}{\lambda_n(0, \pi, \beta)} dt \right)^2 = 0. \end{aligned}$$

Since $\left\{ \frac{\sin \lambda_n(0, \pi, \beta) t}{\lambda_n(0, \pi, \beta)} \right\}_{n=0}^{\infty}$ are the eigenfunctions of the problem $L(0, \pi, \beta)$, then we can extend the function $p(t)$ by zero for $t > x$ and write Parseval's identity in the following form

$$\int_0^{\pi} p^2(t) dt = \int_0^x p^2(t) dt = \sum_{n=0}^{\infty} \frac{1}{a_n(0, \pi, \beta)} \left(\int_0^x p(t) \frac{\sin \lambda_n(0, \pi, \beta) t}{\lambda_n(0, \pi, \beta)} dt \right)^2,$$

and therefore,

$$\sum_{n=0}^{\infty} \frac{1}{a_n} \left(\int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt \right)^2 = 0.$$

Since $a_n > 0$ (see (0.20)), then

$$\int_0^x p(t) \frac{\sin \lambda_n t}{\lambda_n} dt = 0, \quad n \geq 0.$$

In Theorem 1.15, it was proved that the system of functions $\{\sin \lambda_n(q, \pi, \beta)t\}_{n=0}^{\infty}$, if there is no zeros among $\lambda_n(q, \pi, \beta)$, $n = 0, 1, 2, \dots$ (or $\{\sin \lambda_n t\}_{n=0}^{n_0-1} \cup \{t\} \cup \{\sin \lambda_n t\}_{n_0+1}^{\infty}$, if $\lambda_{n_0}(q, \pi, \beta) = 0$) is a Riesz basis in $L^2[0, \pi]$. In fact, there was proved a stronger statement: if the sequence λ_n , $n = 0, 1, 2, \dots$ has the following asymptotic property:

$$\lambda_n = n + \delta_n(\pi, \beta) + O(n^{-1}), \quad \lambda_n^2 \neq \lambda_m^2 \quad (n \neq m), \quad (2.35)$$

then $\{\sin \lambda_n t\}_{n=0}^{\infty}$ if there is no zeros among λ_n , $n = 0, 1, 2, \dots$ (or $\{\sin \lambda_n t\}_{n=0}^{n_0-1} \cup \{t\} \cup \{\sin \lambda_n t\}_{n_0+1}^{\infty}$, if $\lambda_{n_0} = 0$) is a Riesz basis in $L^2[0, \pi]$.

Since our given numbers $\{\lambda_n\}_{n \geq 0}$ (see (0.19)) satisfy asymptotic property (2.35), then $\{\sin \lambda_n t\}_{n=0}^{\infty}$ (or $\{\sin \lambda_n t\}_{n=0}^{n_0-1} \cup \{t\} \cup \{\sin \lambda_n t\}_{n_0+1}^{\infty}$) is a Riesz basis and accordingly is complete in $L^2[0, \pi]$. Hence, we have $p(t) = 0$. This completes the proof. \square

Remark 2.1 *It is noteworthy that, in contrast to the above-mentioned classical results (see [45, 59, 60, 62]), in our case the Levinson's theorem (see [81, 85, 88]) cannot be applied for proving completeness of the system of functions $\{\sin \lambda_n t\}_{n=0}^{\infty}$, because of asymptotic behavior of λ_n .*

Observe that the solution $P(x, t)$ of (2.16) has the same smoothness as $F(x, t)$ (see, e.g. [62, equation (1.5.16) on p. 39]).

Let us define

$$\varphi_{\pi}(x, \mu) \equiv \varphi_{\pi}(x, \lambda^2) := \frac{\sin \lambda x}{\lambda} + \int_0^x P(x, t) \frac{\sin \lambda t}{\lambda} dt \quad (2.36)$$

and

$$q(x) := 2 \frac{d}{dx} P(x, x). \quad (2.37)$$

Lemma 2.3 *The following relations hold*

$$-\varphi_{\pi}''(x, \mu) + q(x) \varphi_{\pi}(x, \mu) = \mu \varphi_{\pi}(x, \mu), \quad (2.38)$$

$$\varphi_{\pi}(0, \mu) = 0, \quad \varphi_{\pi}'(0, \mu) = 1. \quad (2.39)$$

Proof. We split the proof into two cases:

Case I: Let $H'(\cdot) \in AC(0, 2\pi)$, where $H(t)$ defined by (2.3). Differentiating the identity

$$J(x, t) \equiv P(x, t) + F(x, t) + \int_0^x P(x, s) F(s, t) ds \equiv 0, \quad (2.40)$$

we get

$$J_t(x, t) = P_t(x, t) + F_t(x, t) + \int_0^x P(x, s) F_t(s, t) ds = 0, \quad (2.41)$$

$$J_{tt}(x, t) = P_{tt}(x, t) + F_{tt}(x, t) + \int_0^x P(x, s) F_{tt}(s, t) ds = 0, \quad (2.42)$$

$$J_x(x, t) = P_x(x, t) + F_x(x, t) + P(x, x) F(x, t) + \int_0^x P_x(x, s) F(s, t) ds = 0, \quad (2.43)$$

$$\begin{aligned} J_{xx}(x, t) = P_{xx}(x, t) + F_{xx}(x, t) + \frac{dP(x, x)}{dx} F(x, t) + P(x, x) F_x(x, t) + \\ + \frac{\partial P(x, s)}{\partial x} \Big|_{s=x} F(x, t) + \int_0^x P_{xx}(x, s) F(s, t) ds = 0. \end{aligned} \quad (2.44)$$

It is easy to see that $F_{xx}(x, t) = F_{tt}(x, t)$, $F(0, t) = 0$, $F(x, 0) = 0$. Then, the identity (2.40) for $t = 0$ gives $P(x, 0) = 0$.

After integrating (2.42) by parts twice, we obtain

$$\begin{aligned} J_{tt}(x, t) = P_{tt}(x, t) + F_{tt}(x, t) + P(x, x) \frac{\partial F(s, t)}{\partial s} \Big|_{s=x} - \\ - F(x, t) \frac{\partial P(x, s)}{\partial s} \Big|_{s=x} + \int_0^x P_{ss}(x, s) F(s, t) ds = 0. \end{aligned} \quad (2.45)$$

Consider the following identity

$$\begin{aligned} J_{xx}(x, t) - J_{tt}(x, t) - q(x) J(x, t) \equiv P_{xx}(x, t) + \frac{dP(x, x)}{dx} F(x, t) + P(x, x) F_x(x, t) + \\ + \frac{\partial P(x, s)}{\partial x} \Big|_{s=x} F(x, t) + \int_0^x P_{xx}(x, s) F(s, t) ds - P_{tt}(x, t) - \\ - P(x, x) \frac{\partial F(s, t)}{\partial s} \Big|_{s=x} + F(x, t) \frac{\partial P(x, s)}{\partial s} \Big|_{s=x} - \int_0^x P_{ss}(x, s) F(s, t) ds - \\ - q(x) \left(P(x, t) + F(x, t) + \int_0^x P(x, s) F(s, t) ds \right) \equiv 0. \end{aligned} \quad (2.46)$$

Since $q(x) \equiv 2 \frac{dP(x, x)}{dx} \equiv 2 \left(\frac{\partial P(x, s)}{\partial x} \Big|_{s=x} + \frac{\partial P(x, s)}{\partial s} \Big|_{s=x} \right)$, then (2.46) is equivalent to the identity

$$\begin{aligned} J_{xx}(x, t) - J_{tt}(x, t) - q(x) J(x, t) \equiv P_{xx}(x, t) - P_{tt}(x, t) - q(x) P(x, t) + \\ + \int_0^x (P_{xx}(x, s) - P_{ss}(x, s) - q(x) P(x, s)) F(s, t) ds \equiv 0. \end{aligned} \quad (2.47)$$

According to Lemma 2.2, the last equation has only the trivial solution, i.e.

$$P_{xx}(x, t) - P_{tt}(x, t) - q(x)P(x, t) = 0, \quad 0 \leq t < x. \quad (2.48)$$

On one hand, differentiating $\varphi_\pi(x, \mu)$ (see (2.36)) twice, we obtain

$$\varphi'_\pi(x, \mu) = \cos \lambda x + P(x, x) \frac{\sin \lambda x}{\lambda} + \int_0^x P_x(x, t) \frac{\sin \lambda t}{\lambda} dt \quad (2.49)$$

and

$$\begin{aligned} \varphi''_\pi(x, \mu) = & -\lambda \sin \lambda x + \frac{dP(x, x)}{dx} \frac{\sin \lambda x}{\lambda} + P(x, x) \cos \lambda x + \\ & + \left. \frac{\partial P(x, t)}{\partial x} \right|_{t=x} \frac{\sin \lambda x}{\lambda} + \int_0^x P_{xx}(x, t) \frac{\sin \lambda t}{\lambda} dt. \end{aligned} \quad (2.50)$$

On the other hand, integrating $\varphi_\pi(x, \mu)$ by parts twice, we get

$$\mu \varphi_\pi(x, \mu) = \lambda \sin \lambda x - P(x, x) \cos \lambda x + \left. \frac{\partial P(x, t)}{\partial t} \right|_{t=x} \frac{\sin \lambda x}{\lambda} - \int_0^x P_{tt}(x, t) \frac{\sin \lambda t}{\lambda} dt. \quad (2.51)$$

Together with (2.50) and (2.48) this ensures that (2.38) holds and $\varphi_\pi(0, \mu) = 0$. From (2.49) it follows, that $\varphi'_\pi(0, \mu) = 1$, therefore (2.39) also holds.

Case II: Now, let us consider the general case, when (0.19)–(0.22) hold and, according to Lemma 2.1, $H(\cdot) \in AC(0, 2\pi)$. Denote by $\tilde{\varphi}(x, \mu)$ solution of the equation (0.4) (where $q(x) = 2 \frac{d}{dx} P(x, x)$) with initial conditions $\tilde{\varphi}(0, \mu) = 0$, $\tilde{\varphi}'(0, \mu) = 1$. Our goal is to prove that $\tilde{\varphi}(x, \mu) \equiv \varphi_\pi(x, \mu)$.

Choose the sequences of numbers $\{\lambda_{n,(j)}\}_{n \geq 0}$, $\{a_{n,(j)}\}_{n \geq 0}$, $j \geq 1$ of the form

$$\lambda_{n,(j)} = n + \delta_n(\pi, \beta) + \frac{c}{2(n + \delta_n(\pi, \beta))} + l_{n,(j)}, \quad \lambda_{n,(j)}^2 \neq \lambda_{m,(j)}^2 \quad (n \neq m), \quad (2.52)$$

$$a_{n,(j)} = \frac{\pi}{2(n + \delta_n(\pi, \beta))^2} \left(1 + \frac{c_1}{2(n + \delta_n(\pi, \beta))^2} + \frac{2s_{n,(j)}}{\pi(n + \delta_n(\pi, \beta))^2} \right), \quad a_{n,(j)} > 0, \quad (2.53)$$

where c_1 is a constant, the remainders $l_{n,(j)} = o\left(\frac{1}{n^2}\right)$ are such that the functions

$$l'_j(t) = \sum_{n=2}^{\infty} l_{n,(j)} (n + \delta_n(\pi, \beta)) \cos(n + \delta_n(\pi, \beta)) t \quad (2.54)$$

are absolutely continuous on $[0, 2\pi]$ and the remainders $s_{n,(j)} = o(1)$ are such that the functions

$$s'_j(t) = - \sum_{n=2}^{\infty} \frac{s_{n,(j)}}{n + \delta_n(\pi, \beta)} \sin(n + \delta_n(\pi, \beta)) t \quad (2.55)$$

are absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$ (compare with asymptotics of eigenvalues and norming constants from Theorem 1.10 and Theorem 1.11) and

$$\|\Omega_j(t)\|_{W_1^1([0, \pi])} \rightarrow 0 \quad (j \rightarrow \infty), \quad (2.56)$$

where

$$\begin{aligned} \Omega_j(t) := \sum_{n=0}^{\infty} (|\lambda_n - \lambda_{n,(j)}| \sin \lambda_n t + |k_n - k_{n,(j)}| \cos \lambda_n t) + \\ + \sum_{n=0}^{\infty} \left((\lambda_n - \lambda_{n,(j)})^2 + |k_n - k_{n,(j)}| \right), \end{aligned} \quad (2.57)$$

and $W_1^N([0, \pi])$ the Sobolev space of functions $f(x)$, $x \in [0, \pi]$, such that $f^{(i)}(x)$, $i = 0, 1, \dots, N-1$ are absolutely continuous, equipped with the following norm:

$$\|f\|_{W_1^N([0, \pi])} = \sum_{i=0}^N \int_0^{\pi} |f^{(i)}(t)| dt. \quad (2.58)$$

Denote

$$H_j(t) := \sum_{n=0}^{\infty} \left(\frac{1}{a_{n,(j)}} \frac{\cos \lambda_{n,(j)} t - 1}{\lambda_{n,(j)}^2} - \frac{1}{a_n(0, \pi, \beta)} \frac{\cos \lambda_n(0, \pi, \beta) t - 1}{\lambda_n^2(0, \pi, \beta)} \right). \quad (2.59)$$

By the same arguments as in Lemma 2.1 one can prove that $H_j(\cdot) \in AC[0, 2\pi]$ and $H_j'(\cdot) \in AC(0, 2\pi)$. Let $P_j(x, t)$ be the solution of the following Gelfand-Levitan equation

$$P_j(x, t) + F_j(x, t) + \int_0^x P_j(x, s) F_j(s, t) ds = 0, \quad 0 \leq t < x,$$

where $F_j(x, t) = \frac{1}{2} (H_j(|x-t|) - H_j(x+t))$. Take

$$\varphi_{\pi,j}(x, \mu) := \frac{\sin \lambda x}{\lambda} + \int_0^x P_j(x, t) \frac{\sin \lambda t}{\lambda} dt, \quad q_j(x) := 2 \frac{d}{dx} P_j(x, x). \quad (2.60)$$

Since $H_j'(\cdot) \in AC(0, 2\pi)$, then according to Case I, we receive that

$$-\varphi_{\pi,j}''(x, \mu) + q_j(x) \varphi_{\pi,j}(x, \mu) = \mu \varphi_{\pi,j}(x, \mu), \quad (2.61)$$

$$\varphi_{\pi,j}(0, \mu) = 0, \quad \varphi_{\pi,j}'(0, \mu) = 1. \quad (2.62)$$

On one hand, according to (2.56) and Lemma 2.7,

$$\lim_{j \rightarrow \infty} \|H_j(t) - H(t)\|_{W_1^1([0, \pi])} = 0,$$

which implies (by taking into account [62, Lemma 1.5.1 on p. 32]) that

$$\lim_{j \rightarrow \infty} \max_{0 \leq t \leq x \leq \pi} |P_j(x, t) - P(x, t)| = 0 \quad (2.63)$$

and

$$\lim_{j \rightarrow \infty} \|q_j - q\|_{L^1_{\mathbb{R}}} = 0. \quad (2.64)$$

From (2.36), (2.60) and (2.63), it follows that

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} \max_{|\mu| \leq r} |\varphi_{\pi, j}(x, \mu) - \varphi_{\pi}(x, \mu)| = 0.$$

On the other hand, taking into account [38, Theorem 3 on p. 62] (see, also, [61, Theorem 5 on p. 18]), definition of the function $\tilde{\varphi}(x, \mu)$ and (2.64) we arrive at

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} \max_{|\mu| \leq r} |\varphi_{\pi, j}(x, \mu) - \tilde{\varphi}(x, \mu)| = 0.$$

Therefore $\tilde{\varphi}(x, \mu) \equiv \varphi_{\pi}(x, \mu)$. Lemma 2.3 is proved. \square

Using the same methods as in [45, 59, 62], it can be proven that for arbitrary functions $f \in L^2(0, \pi)$

$$\int_0^{\pi} f^2(x) dx = \sum_{n=0}^{\infty} \frac{1}{a_n} \left(\int_0^{\pi} f(t) \varphi_{\pi}(t, \mu_n) dt \right)^2, \quad (2.65)$$

or which is equivalent

$$\int_0^{\pi} f(x) g(x) dx = \sum_{n=0}^{\infty} \frac{1}{a_n} \int_0^{\pi} f(t) \varphi_{\pi}(t, \mu_n) dt \int_0^{\pi} g(t) \varphi_{\pi}(t, \mu_n) dt, \quad (2.66)$$

for every $f, g \in L^2(0, \pi)$.

Lemma 2.4 *The following relation holds*

$$\int_0^{\pi} \varphi_{\pi}(t, \mu_k) \varphi_{\pi}(t, \mu_n) dt = \begin{cases} 0, & n \neq k, \\ a_n, & n = k. \end{cases} \quad (2.67)$$

Proof. Let $f' \in AC[0, \pi]$ and $f(0) = 0$. Consider the following series

$$f^*(x) = \sum_{n=0}^{\infty} c_n \varphi_{\pi}(x, \mu_n), \quad (2.68)$$

where

$$c_n := \frac{1}{a_n} \int_0^\pi f(t) \varphi_\pi(t, \mu_n) dt. \quad (2.69)$$

Using Lemma 2.3 and integration by parts we obtain

$$\begin{aligned} c_n &= \frac{1}{a_n \mu_n} \int_0^\pi f(t) (-\varphi_\pi''(t, \mu_n) + q(t) \varphi_\pi(t, \mu_n)) dt = \\ &= \frac{1}{a_n \mu_n} (f'(\pi) \varphi_\pi(\pi, \mu_n) - f(\pi) \varphi_\pi'(\pi, \mu_n)) + \\ &\quad + \frac{1}{a_n \mu_n} \int_0^\pi (-f''(t) + q(t) f(t)) \varphi_\pi(t, \mu_n) dt. \end{aligned} \quad (2.70)$$

Since $\frac{1}{a_n \mu_n} = O(1)$ (see (0.19) and (0.20)) and

$$\varphi_\pi(x, \mu_n) = \frac{\sin\left(n + \frac{1}{2}\right)x}{n + \frac{1}{2}} + O\left(\frac{1}{n^2}\right), \quad \varphi_\pi'(x, \mu_n) = \cos\left(n + \frac{1}{2}\right)x + O\left(\frac{1}{n}\right)$$

uniformly on $[0, \pi]$, then $c_n = O\left(\frac{1}{n}\right)$. Hence the series (2.68) converges absolutely and uniformly on $[0, \pi]$. According to (2.66) and (2.69), we have

$$\begin{aligned} \int_0^\pi f(x) g(x) dx &= \sum_{n=0}^{\infty} c_n \int_0^\pi g(t) \varphi_\pi(t, \mu_n) dt = \\ &= \int_0^\pi g(t) \sum_{n=0}^{\infty} c_n \varphi_\pi(t, \mu_n) dt = \int_0^\pi g(t) f^*(t) dt. \end{aligned}$$

As $g(x)$ is arbitrary, we get $f^*(x) = f(x)$, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_\pi(x, \mu_n).$$

Now, take $f(x) = \varphi_\pi(x, \mu_k)$ ($k \geq 0$ fixed). The system of functions $\{\varphi_\pi(x, \mu_n)\}_{n \geq 0}$ is minimal in $L^2(0, \pi)$ and therefore

$$c_{nk} := \frac{1}{a_n} \int_0^\pi \varphi_\pi(t, \mu_k) \varphi_\pi(t, \mu_n) dt = \delta_{nk},$$

where δ_{nk} is Kronecker symbol. Lemma 2.4 is proved. \square

Lemma 2.5 For all $n, m \geq 0$

$$\frac{\varphi_\pi'(\pi, \mu_n)}{\varphi_\pi(\pi, \mu_n)} = \frac{\varphi_\pi'(\pi, \mu_m)}{\varphi_\pi(\pi, \mu_m)} = \text{const} \quad (2.71)$$

Proof. It follows from (2.38) that

$$(\varphi_\pi(x, \mu_n) \varphi'_\pi(x, \mu_m) - \varphi'_\pi(x, \mu_n) \varphi_\pi(x, \mu_m))|_0^\pi = (\mu_n - \mu_m) \int_0^\pi \varphi_\pi(x, \mu_n) \varphi_\pi(x, \mu_m) dx.$$

According to (2.67),

$$\varphi_\pi(\pi, \mu_n) \varphi'_\pi(\pi, \mu_m) - \varphi'_\pi(\pi, \mu_n) \varphi_\pi(\pi, \mu_m) = 0. \quad (2.72)$$

Let us show that $\varphi_\pi(\pi, \mu_n) \neq 0$ for all $n \geq 0$. Otherwise from $\varphi_\pi(\pi, \mu_m) = 0$ for a certain m and since $\varphi'_\pi(\pi, \mu_m) \neq 0$, from (2.72) we get that $\varphi_\pi(\pi, \mu_n) = 0$ for all n , which is impossible since

$$\left(n + \frac{1}{2}\right) \varphi_\pi(\pi, \mu_n) = (-1)^n + O\left(\frac{1}{n}\right).$$

Then, dividing (2.72) by $\varphi_\pi(\pi, \mu_n) \varphi_\pi(\pi, \mu_m)$, we arrive at (2.71). Lemma 2.5 is proved. \square

Denote

$$\cot \tilde{\beta} := -\frac{\varphi'_\pi(\pi, \mu_n)}{\varphi_\pi(\pi, \mu_n)}. \quad (2.73)$$

Thus,

$$\varphi_\pi(\pi, \mu_n) \cos \tilde{\beta} + \varphi'_\pi(\pi, \mu_n) \sin \tilde{\beta} = 0, n \geq 0.$$

Together with Lemma 2.3 and Lemma 2.4 this gives that $\{\mu_n\}_{n \geq 0}$ and $\{a_n\}_{n \geq 0}$ are the eigenvalues and the norming constants for the constructed $L(q, \pi, \tilde{\beta})$ problem, respectively.

Remark 2.2 *Note that, on one hand (see (0.19), (1.28) and (1.33))*

$$\cot \tilde{\beta} = \cot \beta + \frac{1}{2} \left(\pi c - \int_0^\pi q(t) dt \right),$$

where $\tilde{\beta}$ and q are defined in (2.73) and (2.37), respectively and β and c determine the sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ (see (0.19), (0.20)).

On the other hand, it is easy to see that $\tilde{\beta}$ may not coincide with β (see section 2.3). In [89] the authors proposed additional conditions (necessary and sufficient) that ensure $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$, when $q \in L^2_{\mathbb{R}}[0, \pi]$, $\alpha, \beta \in (0, \pi)$ problem is considered. We believe that analogous condition can be found in our case as well, but this is outside the scope of the present work.

The Theorem 0.4 is completely proved.

2.3 Implementation of the Algorithm

In this section we are going to present an example of realization of the constructive solution of an inverse Sturm-Liouville problem. To this aim, we will take two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ satisfying the conditions of the Theorem 0.4 and applying the procedure describing in the Section 2.2 to reconstruct the potential q and the parameter $\tilde{\beta}$. Let

$$\sqrt{\mu_n} \equiv \lambda_n = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots, \quad (2.74)$$

$$a_0 = \pi, \quad a_n = \frac{\pi}{2(n + \frac{1}{2})^2}, \quad n = 1, 2, \dots, \quad (2.75)$$

then $\beta = \frac{\pi}{2}$ (see (0.19), (0.20) and (1.33)) and according to (2.15)

$$\begin{aligned} F(x, t) &= \sum_{n=0}^{\infty} \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \frac{2}{\pi} \sin \left(n + \frac{1}{2} \right) x \sin \left(n + \frac{1}{2} \right) t \right) = \\ &= \frac{2}{\pi} \sin \frac{x}{2} \sin \frac{t}{2}. \end{aligned} \quad (2.76)$$

We will seek the solution $P(x, t)$ of the corresponding Gelfand-Levitan equation (see (2.16)) in the form $P(x, t) = a(x) \sin \frac{t}{2}$. After some calculations we find that

$$P(x, t) = \frac{2}{\sin x - x - \pi} \sin \frac{x}{2} \sin \frac{t}{2} \quad (2.77)$$

and

$$q(x) := 2 \frac{d}{dx} P(x, x) = \frac{2 \sin x}{\sin x - x - \pi} - \frac{4(\cos x - 1)}{(\sin x - x - \pi)^2} \sin^2 \frac{x}{2}, \quad (2.78)$$

$$\cot \tilde{\beta} = -\frac{\varphi'_\pi(\pi, \mu_n)}{\varphi_\pi(\pi, \mu_n)} = -P(\pi, \pi) = -\frac{1}{\pi}, \quad (2.79)$$

$$\tilde{\beta} = \operatorname{arccot} \frac{1}{\pi}. \quad (2.80)$$

2.4 Appendix

Here, we prove two lemmas, which play an important role in our analysis.

Lemma 2.6 *Let us denote*

$$T_\beta(x) := \sum_{n=2}^{\infty} \frac{\sin(n + \delta_n(\pi, \beta))x}{n + \delta_n(\pi, \beta)}. \quad (2.81)$$

Then $T_\beta(x)$ and $T'_\beta(x)$ are absolutely continuous functions on arbitrary segment $[a, b] \subset (0, 2\pi)$.

Proof. Denote $t_n = \delta_n(\pi, \beta) - \frac{1}{2}$ and write the general term of the series in (2.81) in the following form

$$\begin{aligned} \frac{\sin(n + \delta_n(\pi, \beta))x}{n + \delta_n(\pi, \beta)} &= \frac{\sin(n + \frac{1}{2} + t_n)x}{n + \frac{1}{2}} - \frac{t_n \sin(n + \frac{1}{2} + t_n)x}{(n + \frac{1}{2})(n + \frac{1}{2} + t_n)} = \\ &= \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} \cos t_n x + \frac{\cos(n + \frac{1}{2})x}{n + \frac{1}{2}} \sin t_n x - \frac{t_n \sin(n + \frac{1}{2})x}{(n + \frac{1}{2})(n + \frac{1}{2} + t_n)} \end{aligned} \quad (2.82)$$

It follows from (1.33), that

$$t_n = \frac{\cot \beta}{\pi(n + \frac{1}{2})} + u_n, \quad \cos t_n x = 1 + v_n(x), \quad \sin t_n x = t_n x + w_n(x),$$

where $u_n = O\left(\frac{1}{n^2}\right)$ and $v_n(x) = O\left(\frac{1}{n^2}\right)$, $w_n(x) = O\left(\frac{1}{n^3}\right)$ uniformly on $[0, 2\pi]$ and therefore (2.82) can be written in the following way:

$$\begin{aligned} \frac{\sin(n + \delta_n(\pi, \beta))x}{n + \delta_n(\pi, \beta)} &= \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} + v_n(x) \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} + \frac{x \cot \beta}{\pi} \frac{\cos(n + \frac{1}{2})x}{(n + \frac{1}{2})^2} + \\ &+ x u_n \frac{\cos(n + \frac{1}{2})x}{n + \frac{1}{2}} + w_n(x) \frac{\cos(n + \frac{1}{2})x}{n + \frac{1}{2}} - \frac{t_n \sin(n + \frac{1}{2})x}{(n + \frac{1}{2})(n + \frac{1}{2} + t_n)}. \end{aligned}$$

It is easy to see that

$$T_\beta(x) = T_1(x) + T_{2\beta}(x) + T_{3\beta}(x),$$

where

$$\begin{aligned} T_1(x) &= \sum_{n=2}^{\infty} \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}}, \\ T_{2\beta}(x) &= \frac{x \cot \beta}{\pi} \sum_{n=2}^{\infty} \frac{\cos(n + \frac{1}{2})x}{(n + \frac{1}{2})^2}, \\ T_{3\beta}(x) &= \sum_{n=2}^{\infty} \left(v_n(x) \frac{\sin(n + \frac{1}{2})x}{n + \frac{1}{2}} + x u_n \frac{\cos(n + \frac{1}{2})x}{n + \frac{1}{2}} + \right. \\ &\quad \left. + w_n(x) \frac{\cos(n + \frac{1}{2})x}{n + \frac{1}{2}} - \frac{t_n \sin(n + \frac{1}{2})x}{(n + \frac{1}{2})(n + \frac{1}{2} + t_n)} \right). \end{aligned}$$

From [75, formulae 37 and 38 on p. 578], we get that

$$T_1(x) = \frac{\pi}{2} - 2 \sin \frac{x}{2} - \frac{2}{3} \sin \frac{3x}{2}, \quad 0 < x < 2\pi$$

and

$$T_{2\beta}(x) = \frac{x \cot \beta}{\pi} \left(\frac{\pi^2 - 2\pi x}{2} - 4 \cos \frac{x}{2} - \frac{4}{9} \cos \frac{3x}{2} \right), \quad 0 \leq x \leq 2\pi$$

are infinitely differentiable functions on corresponding domains. The estimates for u_n , $v_n(x)$ and $w_n(x)$ ensure the absolute continuity of the functions $T_{3\beta}$ and $T'_{3\beta}$. This completes the proof. \square

Lemma 2.7 *Let us given the functions $\Omega_j(t)$, $H_j(t)$ and $H(t)$ defined in (2.57), (2.59) and (2.3), respectively. Then, there are C_1 and C_2 positive numbers such that*

$$\max_{0 \leq t \leq 2\pi} |H_j(t) - H(t)| \leq C_1 \max_{0 \leq t \leq 2\pi} |\Omega_j(t)|, \quad (2.83)$$

$$\|H_j(t) - H(t)\|_{W_1^1([0, \pi])} \leq C_2 \|\Omega_j(t)\|_{W_1^1([0, \pi])}. \quad (2.84)$$

Proof. Using (2.59) and (2.3) we write down $H_j(t) - H(t)$ and $H'_j(t) - H'(t)$ in the following forms:

$$\begin{aligned} H_j(t) - H(t) &= \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{a_{n,(j)}} \frac{\cos \lambda_{n,(j)}t - 1}{\lambda_{n,(j)}^2} - \frac{1}{a_n} \frac{\cos \lambda_n t - 1}{\lambda_n^2} \right) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_{n,(j)}t - 1}{k_{n,(j)}} - \frac{\cos \lambda_n t - 1}{k_n} \right) = \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{k_{n,(j)}} (\cos \lambda_{n,(j)}t - \cos \lambda_n t) + \left(\frac{1}{k_{n,(j)}} - \frac{1}{k_n} \right) (\cos \lambda_n t - 1) \right) = \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{k_{n,(j)}} \sin \frac{(\lambda_n - \lambda_{n,(j)})t}{2} \sin \frac{(\lambda_n + \lambda_{n,(j)})t}{2} + \frac{k_n - k_{n,(j)}}{k_n k_{n,(j)}} (\cos \lambda_n t - 1) \right) = \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{k_{n,(j)}} \sin \frac{(\lambda_n - \lambda_{n,(j)})t}{2} \sin \left(\lambda_n t - \frac{(\lambda_n - \lambda_{n,(j)})t}{2} \right) + \right. \\ &\quad \left. + \frac{k_n - k_{n,(j)}}{k_n k_{n,(j)}} (\cos \lambda_n t - 1) \right) = \sum_{n=0}^{\infty} \left(\frac{1}{k_{n,(j)}} \sin (\lambda_n - \lambda_{n,(j)})t \sin \lambda_n t - \right. \\ &\quad \left. - \frac{2}{k_{n,(j)}} \sin^2 \frac{(\lambda_n - \lambda_{n,(j)})t}{2} \cos \lambda_n t + \frac{k_n - k_{n,(j)}}{k_n k_{n,(j)}} (\cos \lambda_n t - 1) \right), \quad (2.85) \end{aligned}$$

$$\begin{aligned} H'_j(t) - H'(t) &= \sum_{n=0}^{\infty} \left(\frac{\lambda_n - \lambda_{n,(j)}}{k_{n,(j)}} \cos (\lambda_n - \lambda_{n,(j)})t \sin \lambda_n t + \right. \\ &\quad \left. + \frac{\lambda_n}{k_{n,(j)}} \sin (\lambda_n - \lambda_{n,(j)})t \cos \lambda_n t \right) - \sum_{n=0}^{\infty} \left(\frac{\lambda_n - \lambda_{n,(j)}}{k_{n,(j)}} \sin (\lambda_n - \lambda_{n,(j)})t \cos \lambda_n t - \right. \\ &\quad \left. - \frac{2\lambda_n}{k_{n,(j)}} \sin^2 \frac{(\lambda_n - \lambda_{n,(j)})t}{2} \sin \lambda_n t \right) - \sum_{n=0}^{\infty} \frac{k_n - k_{n,(j)}}{k_n k_{n,(j)}} \lambda_n \sin \lambda_n t. \quad (2.86) \end{aligned}$$

From (2.85) and (2.86) it is easy to get the following estimates:

$$|H_j(t) - H(t)| \leq C_1 \Omega_j(t), \quad (2.87)$$

$$|H'_j(t) - H'(t)| \leq C_1 (\Omega_j(t) + |\Omega'_j(t)|). \quad (2.88)$$

These prove the lemma. \square

Lemma 2.8 *Let us denote*

$$f(\delta) := \delta - \frac{1}{\pi} \arccos \frac{\cos \alpha}{\sqrt{(n+\delta)^2 \sin^2 \alpha + \cos^2 \alpha}} + \frac{1}{\pi} \arccos \frac{\cos \beta}{\sqrt{(n+\delta)^2 \sin^2 \beta + \cos^2 \beta}}. \quad (2.89)$$

Then, for each fixed $n = n_0 \geq 2$, $\alpha = \alpha_0 \in (0, \pi]$, $\beta = \beta_0 \in [0, \pi)$ the equation

$$f(\delta) = 0 \quad (2.90)$$

has a unique solution.

Proof. Let $n = n_0 \geq 2$, $\alpha = \alpha_0 \in (0, \pi]$, $\beta = \beta_0 \in [0, \pi)$ be fixed. First we prove that the equation (2.90) has a solution. To this end, recall that $0 \leq \frac{1}{\pi} \arccos x \leq 1$, for all $x \in [-1, 1]$ and calculate the values of the function f at the points -1 and 1 :

$$\begin{aligned} f(-1) = -1 - \frac{1}{\pi} \arccos \frac{\cos \alpha_0}{\sqrt{(n_0-1)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0}} + \\ + \frac{1}{\pi} \arccos \frac{\cos \beta_0}{\sqrt{(n_0-1)^2 \sin^2 \beta_0 + \cos^2 \beta_0}} \leq 0, \end{aligned} \quad (2.91)$$

$$\begin{aligned} f(1) = 1 - \frac{1}{\pi} \arccos \frac{\cos \alpha_0}{\sqrt{(n_0+1)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0}} + \\ + \frac{1}{\pi} \arccos \frac{\cos \beta_0}{\sqrt{(n_0+1)^2 \sin^2 \beta_0 + \cos^2 \beta_0}} \geq 0, \end{aligned} \quad (2.92)$$

Since $f(\delta)$ is a continuous function, then it follows from (2.91), (2.92) and intermediate value theorem, that there is a $\tilde{\delta} \in [-1, 1]$, such that $f(\tilde{\delta}) = 0$. On the other hand, it is easy to see that the equation (2.90) has no solution outside the segment $[-1, 1]$.

Now, let us prove the uniqueness of the solution of the equation (2.90). To this end,

calculate the first derivative of the function f :

$$\begin{aligned}
f'(\delta) &= 1 + \frac{\cos \alpha_0}{\pi \sqrt{1 - \frac{\cos^2 \alpha_0}{(n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0}}} \left(\frac{1}{\sqrt{(n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0}} \right)' - \\
&\quad - \frac{\cos \beta_0}{\pi \sqrt{1 - \frac{\cos^2 \beta_0}{(n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0}}} \left(\frac{1}{\sqrt{(n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0}} \right)' = 1 - \\
&\quad - \frac{\cos \alpha_0}{\pi} \frac{\sqrt{(n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0}}{(n_0 + \delta) \sin \alpha_0} \frac{(n_0 + \delta) \sin^2 \alpha_0}{\sqrt{((n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0)^3}} + \\
&\quad + \frac{\cos \beta_0}{\pi} \frac{\sqrt{(n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0}}{(n_0 + \delta) \sin \beta_0} \frac{(n_0 + \delta) \sin^2 \beta_0}{\sqrt{((n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0)^3}} = \\
&= 1 - \frac{1}{2\pi} \frac{\sin 2\alpha_0}{(n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0} + \frac{1}{2\pi} \frac{\sin 2\beta_0}{(n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0} \quad (2.93)
\end{aligned}$$

Since $(n_0 + \delta)^2 \sin^2 \alpha_0 + \cos^2 \alpha_0 \geq 1$ and $(n_0 + \delta)^2 \sin^2 \beta_0 + \cos^2 \beta_0 \geq 1$ for all $\delta \in [-1, 1]$ ($n_0 \geq 2$, $\alpha_0 \in (0, \pi]$, $\beta_0 \in [0, \pi)$), then from (2.93), it follows that $f'(\delta) \geq 1 - \frac{1}{\pi} > 0$, for all $\delta \in [-1, 1]$. This implies that there are no other solutions of the equation (2.90) except $\tilde{\delta}$. Indeed, if there were another solution $\tilde{\tilde{\delta}}$ of the equation (2.90), then according to Roll's theorem, there is $\hat{\delta} \in (\tilde{\delta}, \tilde{\tilde{\delta}})$, such that $f'(\hat{\delta}) = 0$. Lemma 2.8 is proved. \square

Zeros of Eigenfunctions and Sturm Oscillation Theorem

In this chapter we study the dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters that determining the boundary conditions. As a corollary, we obtain Sturm oscillation theorem, which states that the n -th eigenfunction has n zeros.

3.1 On the “movement” of the zeros of eigenfunctions

In the papers [69, 90] it was introduced the concept of the eigenvalues function (EVF) of a family of Sturm-Liouville operators $\{L(q, \alpha, \beta); \alpha \in (0, \pi], \beta \in [0, \pi)\}$. For fixed q it is a function of two variables γ and δ ($\gamma = \alpha + \pi n \in (0, \infty)$, $\delta = \beta - \pi m \in (-\infty, \pi)$) determined through the eigenvalues $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, by the following formula:

$$\mu(\gamma, \delta) = \mu(\alpha + \pi n, \beta - \pi m) := \mu_{n+m}(q, \alpha, \beta), \quad n, m = 0, 1, 2, \dots \quad (3.1)$$

It is proved that this function is analytic with respect to γ and δ . It is strictly increasing by γ and strictly decreasing by δ :

$$\frac{\partial \mu_n(q, \alpha, \beta)}{\partial \alpha} = \frac{1}{a_n} > 0, \quad \frac{\partial \mu_n(q, \alpha, \beta)}{\partial \beta} = -\frac{1}{b_n} < 0.$$

It is also known (see, e.g. [47]) that every nontrivial solution $y(x, \mu)$ of the equation (0.4) may have only simple zeros (if $y(x_0, \mu) = 0$, then $y'(x_0, \mu) \neq 0$), and every solution $y(x, \mu)$ is a continuously differentiable function with respect to the totality of variables x and μ . Therefore, by applying the implicit function theorem (see, e.g. [91, p. 452]), we get that the zeros of the solution $y(x, \mu)$ are the continuously differentiable functions with respect to μ . Since the function $x = x(\mu)$, such that the identity $y(x(\mu), \mu) \equiv 0$ is true for all μ from some interval (a, b) , is called the solution of the equation $y(x, \mu) = 0$, then differentiating the last

identity with respect to μ we obtain:

$$\frac{dy(x(\mu), \mu)}{d\mu} = \frac{\partial y(x(\mu), \mu)}{\partial x} \frac{dx(\mu)}{d\mu} + \frac{\partial y(x(\mu), \mu)}{\partial \mu} \equiv 0, \quad \mu \in (a, b). \quad (3.2)$$

Let us denote $\frac{\partial y(x, \mu)}{\partial \mu} := \dot{y}(x, \mu)$ and write the identity (3.2) in the following form:

$$\frac{dx(\mu)}{d\mu} = \dot{x}(\mu) = -\frac{\dot{y}(x(\mu), \mu)}{y'(x(\mu), \mu)}, \quad \mu \in (a, b). \quad (3.3)$$

On the other hand, let us write down the fact that $y(x, \mu)$ is the solution of the equation (0.4), i.e.

$$-y''(x, \mu) + q(x)y(x, \mu) \equiv \mu y(x, \mu), \quad 0 < x < \pi, \quad \mu \in \mathbb{C}, \quad (3.4)$$

and differentiating this identity with respect to μ we receive:

$$-\dot{y}''(x, \mu) + q(x)\dot{y}(x, \mu) \equiv y(x, \mu) + \mu\dot{y}(x, \mu). \quad (3.5)$$

Multiplying (3.4) by \dot{y} , (3.5) by y and subtracting from the second obtained identity the first one, we get

$$y''(x, \mu)\dot{y}(x, \mu) - \dot{y}''(x, \mu)y(x, \mu) \equiv y^2(x, \mu), \quad 0 < x < \pi, \quad \mu \in \mathbb{C},$$

i.e.

$$\frac{d}{dx} [y'(x, \mu)\dot{y}(x, \mu) - \dot{y}'(x, \mu)y(x, \mu)] \equiv y^2(x, \mu). \quad (3.6)$$

If we integrate this identity with respect to x from 0 to a ($0 \leq a \leq \pi$), then

$$y'(a, \mu)\dot{y}(a, \mu) - \dot{y}'(a, \mu)y(a, \mu) - y'(0, \mu)\dot{y}(0, \mu) + \dot{y}'(0, \mu)y(0, \mu) = \int_0^a y^2(x, \mu) dx, \quad (3.7)$$

and if we integrate with respect to x from a to π , then we get

$$y'(\pi, \mu)\dot{y}(\pi, \mu) - \dot{y}'(\pi, \mu)y(\pi, \mu) - y'(a, \mu)\dot{y}(a, \mu) + \dot{y}'(a, \mu)y(a, \mu) = \int_a^\pi y^2(x, \mu) dx. \quad (3.8)$$

Now, as $y(x, \mu)$ let us take $y = \varphi(x, \mu, \alpha, q)$ – the solution of the equation (0.4), satisfying the following initial conditions:

$$\varphi(0, \mu, \alpha, q) = \sin \alpha, \quad \varphi'(0, \mu, \alpha, q) = -\cos \alpha. \quad (3.9)$$

It is easy to see that eigenfunctions of the problem $L(q, \alpha, \beta)$ are obtained from the solution $\varphi(x, \mu, \alpha, q)$ at $\mu = \mu_n(q, \alpha, \beta)$ (here we use (3.1)), i.e.

$$\begin{aligned} \varphi_n(x, q, \alpha, \beta) &:= \varphi_n(x) = \varphi(x, \mu_n(q, \alpha, \beta), \alpha, q) = \varphi(x, \mu(\alpha + \pi n, \beta), \alpha, q) = \\ &= \varphi(x, \mu(\alpha, \beta - \pi n), \alpha, q) = \varphi(x, \mu(\alpha, \delta), \alpha, q)|_{\delta=\beta-\pi n} = \varphi(x, \mu(\alpha, \delta))|_{\delta=\beta-\pi n} = \\ &= \varphi(x, \mu(\alpha, \beta - \pi n)). \end{aligned} \quad (3.10)$$

Let $0 \leq x_n^0 < x_n^1 < \dots < x_n^m \leq \pi$ be the zeros of the eigenfunction $\varphi_n(x, q, \alpha, \beta) = \varphi(x, \mu(\alpha, \beta - \pi n))$, i.e. $\varphi_n(x_n^k, q, \alpha, \beta) = \varphi(x_n^k, \mu(\alpha, \beta - \pi n)) = 0$, $k = 0, 1, \dots, m$.

Let q, α, n be fixed. We consider the following questions:

- a) How are the zeros $x_n^k = x_n^k(\beta)$, $k = 0, 1, \dots, m$ changing, when β is changing on $[0, \pi]$?
- b) How many zeros the n -th eigenfunction $\varphi_n(x, q, \alpha, \beta)$ has, i.e. what is equal m ?

By taking $y = \varphi_n(x)$ and $a = x_n^k$ ($k = 0, 1, \dots, m$) in (3.7), we receive

$$\varphi_n'(x_n^k) \dot{\varphi}_n(x_n^k) - \dot{\varphi}_n'(x_n^k) \varphi_n(x_n^k) - \varphi_n'(0) \dot{\varphi}_n(0) + \dot{\varphi}_n'(0) \varphi_n(0) = \int_0^{x_n^k} \varphi_n^2(x) dx. \quad (3.11)$$

Since the initial conditions (3.9) should be held for all $\mu \in \mathbb{C}$, then $\dot{\varphi}_n(0) = 0$ and $\dot{\varphi}_n'(0) = 0$.

Also, taking into account that $\varphi_n(x_n^k) = 0$, from (3.11) we get

$$\varphi_n'(x_n^k) \dot{\varphi}_n(x_n^k) = \int_0^{x_n^k} \varphi_n^2(x) dx. \quad (3.12)$$

Since the zeros of the solutions are simple, then $\varphi_n'(x_n^k) \neq 0$, and therefore, from (3.12) the following equality implies :

$$\frac{\dot{\varphi}_n(x_n^k)}{\varphi_n'(x_n^k)} = \frac{1}{(\varphi_n'(x_n^k))^2} \int_0^{x_n^k} \varphi_n^2(x) dx. \quad (3.13)$$

Now, from (3.3), we obtain

$$\dot{x}_n^k(\mu_n) = \left. \frac{dx_n^k(\mu)}{d\mu} \right|_{\mu=\mu_n} = -\frac{\dot{\varphi}_n(x_n^k)}{\varphi_n'(x_n^k)} = -\frac{1}{(\varphi_n'(x_n^k))^2} \int_0^{x_n^k} \varphi_n^2(x) dx, \quad (3.14)$$

i.e. zeros $x_n^k(\mu_n)$, $k = 0, 1, \dots, m$ of the eigenfunction $\varphi_n(x)$ are decreasing if the eigenvalue $\mu_n(q, \alpha, \beta)$ is increasing, which in its turn means that

$$\dot{x}_n^k(\mu_n(q, \alpha, \beta)) \leq 0. \quad (3.15)$$

Let us note, that the equality $\dot{x}_n^k(\mu_n) = 0$ may occur only at $x_n^k = 0$, i.e., when $x = 0$ is a zero of the eigenfunction $\varphi_n(x)$, and it is so at $\alpha = \pi$ ($\gamma = \pi l$, $l = 1, 2, \dots$).

Meanwhile, in the inequality $\dot{x}_n^k(\mu_n) = \dot{x}_n^k(\mu_n(q, \alpha, \beta)) \leq 0$, the variable μ_n can be changed depending on q , α and β . More precisely, if for some change in these three variables $\mu_n(q, \alpha, \beta)$ increases, then the zeros of the eigenfunction $\varphi_n(x)$ are moving to the left, and if $\mu_n(q, \alpha, \beta)$ decreases, then the zeros of the eigenfunction $\varphi_n(x)$ are moving to the right.

In the work [61], it was proved for $q \in L_{\mathbb{R}}^2[0, \pi]$ that the number of zeros of the n -th eigenfunctions of the problems $L(q, \pi, 0)$ and $L(0, \pi, 0)$ are equal. But it is easy to see that the same proof remains true for $q \in L_{\mathbb{R}}^1[0, \pi]$.

It is easy to calculate that the eigenvalues of the problem $L(0, \pi, 0)$ are $\mu_n(0, \pi, 0) = (n+1)^2$ and the eigenfunctions are

$$\varphi_n(x) = \varphi(x, \mu_n(0, \pi, 0), \pi, 0) = \varphi(x, (n+1)^2, \pi, 0) = \frac{\sin(n+1)x}{n+1}, \quad n = 0, 1, 2, \dots \quad (3.16)$$

The zeros of this eigenfunction are $x_n^k = \frac{\pi k}{n+1}$, $k = 0, 1, \dots, n+1$, i.e. the n -th eigenfunction of the problem $L(0, \pi, 0)$ has $n+2$ zeros in $[0, \pi]$, two of which are 0 and π , and n zeros in $(0, \pi)$.

Thereby, the n -th eigenfunction $\varphi(x, q, \pi, 0)$ of the problem $L(q, \pi, 0)$ has two zeros at the endpoints of $[0, \pi]$, i.e. $x_n^0(q, \pi, 0) = 0$, $x_n^{n+1}(q, \pi, 0) = \pi$ and also n zeros in the interval $(0, \pi)$.

Along with increasing β from 0 to π the eigenvalue $\mu_n(q, \pi, 0)$ is continuously (with respect to β) decreasing from $\mu_n(q, \pi, 0)$ to $\mu_n(q, \pi, \pi) = \mu(\pi, \pi - \pi n) = \mu(\pi, 0 - (n-1)\pi) = \mu_{n-1}(q, \pi, 0)$ and, according to (3.14), the zeros of the function $\varphi_n(x, q, \pi, \beta)$ are increasing, i.e. are moving to the right (all but leftmost zero $x_n^0 = 0$). In particular, the rightmost zero $x_n^{n+1} = \pi$, by moving to the right, leaves the segment $[0, \pi]$ and in $[0, \pi]$ remains $n+1$ zeros (one $x_n^0 = 0$ and n zeros in the interval $(0, \pi)$). And the previous zero reaches π when the relation $\varphi_n(\pi) = \varphi(\pi, \mu_n(q, \pi, \beta), \pi, q) = c_n \psi_n(\pi) \sin \beta = 0$ again occurs (see below (3.17) and (3.20)), and this is possible only when β reaches π (and $\mu_n(q, \pi, \beta)$, by decreasing, reaches $\mu_n(q, \pi, \pi) = \mu_{n-1}(q, \pi, 0)$). Then the eigenfunction $\varphi_n(x) = \varphi(x, \mu_n(q, \pi, \beta), \beta, q)$ will smoothly transform to the eigenfunction

$$\varphi(x, \mu_n(q, \pi, \pi), \pi, q) = \varphi(x, \mu_{n-1}(q, \pi, 0), \pi, q)$$

which has $n + 1$ zeros in $[0, \pi]$, two out of which are the endpoints 0 and π , and $n - 1$ zeros are in $(0, \pi)$. Thus, the oscillation theorem is proved for all $L(q, \pi, \beta)$, $\beta \in [0, \pi)$.

Now, as $y(x, \mu)$ let us take $\psi(x, \mu, \beta, q)$ – the solution of the equation (0.4) satisfying the following initial conditions:

$$\psi(\pi, \mu, \beta, q) = \sin \beta, \quad \psi'(\pi, \mu, \beta, q) = -\cos \beta. \quad (3.17)$$

It is easy to see that the eigenvalues $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, \dots$, of the problem $L(q, \alpha, \beta)$ are the zeros of the entire function

$$\Psi(\mu) = \Psi(\mu, \alpha, \beta, q) = \psi(0, \mu, \beta, q) \cos \alpha + \psi'(0, \mu, \beta, q) \sin \alpha, \quad (3.18)$$

and the eigenfunctions, corresponding to these eigenvalues, are obtained by the formula

$$\psi_n(x) = \psi(x, \mu_n(q, \alpha, \beta), \beta, q), \quad n = 0, 1, \dots \quad (3.19)$$

Since all eigenvalues μ_n are simple, then eigenfunctions $\varphi_n(x)$ and $\psi_n(x)$ corresponding to the same eigenvalue μ_n are linearly dependent, i.e. there exist the constants $c_n = c_n(q, \alpha, \beta)$, $n = 0, 1, \dots$, such that

$$\varphi_n(x) = c_n \psi_n(x), \quad n = 0, 1, \dots \quad (3.20)$$

This implies that $\varphi_n(x, q, \alpha, \beta)$ and $\psi_n(x, q, \alpha, \beta)$ have equal number of zeros.

Let $0 \leq x_n^m < x_n^{m-1} < \dots < x_n^0 \leq \pi$ be the zeros of the eigenfunction $\psi_n(x, q, \alpha, \beta)$ i.e. $\psi_n(x_n^k, q, \alpha, \beta) = 0$, $k = 0, 1, \dots, m$.

By taking in the identity (3.8) $y = \psi_n(x) = \psi(x, \mu_n(q, \alpha, \beta), \beta, q)$, we receive

$$\begin{aligned} \psi_n'(\pi) \dot{\psi}_n(\pi) - \dot{\psi}_n'(\pi) \psi_n(\pi) - \psi_n'(x_n^k) \dot{\psi}_n(x_n^k) + \\ + \dot{\psi}_n'(x_n^k) \psi_n(x_n^k) = \int_{x_n^k}^{\pi} \psi_n^2(x) dx. \end{aligned} \quad (3.21)$$

From (3.17) we get that $\dot{\psi}_n(\pi) = 0$ and $\dot{\psi}_n'(\pi) = 0$. And since $\psi_n(x_n^k) = 0$, then the equality (3.21) gets the form

$$-\psi_n'(x_n^k) \dot{\psi}_n(x_n^k) = \int_{x_n^k}^{\pi} \psi_n^2(x) dx. \quad (3.22)$$

As all zeros x_n^k are simple, i.e. $\psi'_n(x_n^k) \neq 0$, dividing both sides of the last equality by $(\psi'_n(x_n^k))^2$, we obtain

$$\frac{\dot{\psi}_n(x_n^k)}{\psi'_n(x_n^k)} = -\frac{1}{(\psi'_n(x_n^k))^2} \int_{x_n^k}^{\pi} \psi_n^2(x) dx. \quad (3.23)$$

Now from (3.3), by taking $y = \psi_n(x)$, $x = x_n^k$, we have

$$\dot{x}_n^k(\mu_n) = \left. \frac{dx_n^k(\mu)}{d\mu} \right|_{\mu=\mu_n} = -\frac{\dot{\psi}_n(x_n^k)}{\psi'_n(x_n^k)} = \frac{1}{(\psi'_n(x_n^k))^2} \int_{x_n^k}^{\pi} \psi_n^2(x) dx \geq 0, \quad (3.24)$$

i.e. the zeros $x_n^k(\mu_n)$, $k = 0, 1, \dots, m$ of the eigenfunction $\psi_n(x)$ are increasing, if the eigenvalue μ_n is increasing. Note that the equality $\dot{x}_n^k(\mu_n) = 0$ is possible only when $x_n^0(\mu_n) = \pi$, but it holds when $\beta = 0$ ($\delta = -\pi l$, $l = 0, 1, 2, \dots$).

While studying the dependence of the zeros of eigenfunctions on α , it is convenient to use formula (3.24), because the eigenfunctions $\psi_n(x)$ have fixed values $\psi_n(\pi, \mu, \beta) = \sin \beta$ and $\psi'_n(\pi, \mu, \beta) = -\cos \beta$ for all $\mu \in \mathbb{C}$, i.e. all $\psi_n(x)$ satisfy the initial conditions $\psi_n(\pi) = \sin \beta$ and $\psi'_n(\pi) = -\cos \beta$, which means that from the endpoint π of the segment $[0, \pi]$ (when changing α) new zeros can neither enter nor leave (neither appear nor disappear). Thus, with increasing α , the eigenvalues $\mu_n(q, \alpha, \beta)$ (with fixed q and β) are increasing and according to (3.24) (i.e. $\dot{x}_n^k(\mu_n) \geq 0$) the zeros of the eigenfunction $\psi_n(x)$ are moving to the right (i.e. are increasing). Wherein, the values $\psi(\pi) = \sin \beta$ and $\psi'(\pi) = -\cos \beta$ are not changed, the number of the zeros increase, and these zeros can neither “collide” nor “split” as they are simple. That’s why new zeros can appear only entering through 0– left endpoint of the segment $[0, \pi]$ and moving to the right (and “condensing” respectively). And new zeros are entered through 0– left endpoint of the segment $[0, \pi]$ only when $\psi_n(0) = 0$, but since $\psi_n(0) = c_n \varphi_n(0) = c_n \sin \alpha$ ($c_n \neq 0$), then the equality $\psi_n(0) = 0$ is possible only when $\sin \alpha = 0$, i.e. in our notations only when $\alpha = \pi$ (and because $\mu_n(q, 0, \beta) = \mu(0 + \pi n, \beta) = \mu(\pi + (n-1)\pi, \beta) = \mu_{n-1}(q, \pi, \beta)$) or when $\alpha = 0$.

Hence, when $\alpha = \pi$, the eigenfunction $\psi_n(x, q, \pi, \beta)$ as well as $\varphi_n(x, q, \pi, \beta)$ has n zeros in $(0, \pi)$ and one zero at $x = 0$ left endpoint. Wherein, $\psi_n(x, q, \pi, \beta) = \psi(x, \mu_n(q, \pi, \beta), \beta, q) = \psi(x, \mu_{n+1}(q, 0, \beta), \beta, q) = \psi_{n+1}(x, q, 0, \beta)$. With increasing α from 0 to π the eigenvalue $\mu_{n+1}(q, 0, \beta)$ is increasing (continuously with respect to α) to $\mu_{n+1}(q, \pi, \beta)$, and the leftmost

zero $x = 0$ by moving to the right, appears in $(0, \pi)$, i.e. there are $n + 1$ zeros of the eigenfunction $\psi_n(x, q, \alpha, \beta)$ in $(0, \pi)$ (and another fixed zero $x = \pi$, if $\beta = 0$). A new zero will appear at $x = 0$ left endpoint, when α will reach $\alpha = \pi$.

Thus, we obtain the following oscillation theorem:

Theorem 3.1 *The eigenfunctions of the problem $L(q, \alpha, \beta)$ corresponding to the n -th eigenvalue $\mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \dots$, have exactly n zeros in $(0, \pi)$. All these zeros are simple. If $\alpha = \pi$ and $\beta = 0$, then the n -th eigenfunction has $n + 2$ zeros in $[0, \pi]$, and if either $\alpha = \pi$, $\beta \in (0, \pi)$ or $\beta = 0$, $\alpha \in (0, \pi)$, then the n -th eigenfunction has $n + 1$ zeros in $[0, \pi]$.*

Oscillation properties of the solutions of the problem $L(q, \alpha, \beta)$ (Sturm theory), the studies of which initiated by Sturm in [1, 2], outlined in monographic literature (see, e.g. [39, 47, 92]) for continuous q . In the recent years, the study was mostly focused on the cases when q is bounded or $q \in L^2_{\mathbb{R}}[0, \pi]$ (see, e.g. [61, 93]), but in many studies (see [94] and references therein) implicitly assumes that Sturm's oscillation theorem (that the n -th eigenfunction has n zeros) is also true for $q \in L^1_{\mathbb{R}}[0, \pi]$, although the rigorous proof is not available in the literature.

Our oscillation theorem is true for all $q \in L^1_{\mathbb{R}}[0, \pi]$.

Conclusion

The main results of this work are:

- 1) The necessary and sufficient conditions are found for two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectrum and the norming constants of a problem $L(q, \pi, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and some $\beta \in (0, \pi)$;
- 2) Uniform convergence of the expansion of an absolutely continuous function for eigenfunctions of the Sturm-Liouville problems $L(q, \pi, \beta)$, $\beta \in (0, \pi)$ and $L(q, \alpha, 0)$, $\alpha \in (0, \pi)$, with summable potential $q \in L^1_{\mathbb{R}}[0, \pi]$ is proved;
- 3) More precise asymptotic formula for the eigenvalues of the problem $L(q, \pi, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and $\beta \in (0, \pi)$ is proved;
- 4) New and more precise asymptotic formulae for the norming constants of the problem $L(q, \alpha, \beta)$, with $q \in L^1_{\mathbb{R}}[0, \pi]$ and $(\alpha, \beta) \in (0, \pi) \times [0, \pi)$ are derived;
- 5) Riesz basicity of the systems $\{\cos \sqrt{\mu_n}x\}_{n=0}^{\infty}$ and $\{\sin \sqrt{\mu_n}x\}_{n=0}^{\infty}$ in $L^2[0, \pi]$ is investigated;
- 6) An analogue of the Gelfand-Levitan equation for the case $q \in L^1_{\mathbb{R}}[0, \pi]$, $\alpha = \pi$, $\beta \in (0, \pi)$ is derived;
- 7) The existence and uniqueness of the solution of the analogue Gelfand-Levitan equation are proved;
- 8) An efficient algorithm (analogue of the Gelfand-Levitan algorithm) for the reconstruction of the Sturm-Liouville problem $L(q, \pi, \beta)$, $q \in L^1_{\mathbb{R}}[0, \pi]$, $\beta \in (0, \pi)$ is provided;
- 9) The dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters that determining the boundary conditions is studied. Sturm oscillation theorem for $q \in L^1_{\mathbb{R}}[0, \pi]$ is obtained.

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Author's Publications on the Topic of the Thesis

- [AP1] A.A. Pahlevanyan. Convergence of expansions for eigenfunctions and asymptotics of the spectral data of the Sturm-Liouville problem (in Russian). *Izv. Nats. Akad. Nauk Armenii Mat.*, 52(6):77–90, 2017.
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anniversary of foundation of Armenian National Academy of Sciences, August 24–31, 2013, Tsaghkadzor, Armenia, pp. 76–77.

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