ԵՐԵՎԱՆԻ ՊԵՏԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ

Ավետիք Արայի Փահլևանյան

Սպեկտրալ տեսության որոշ հարցեր

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մասնագիտությամբ ֆիզիկամաթեմատիկական գիտությունների
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ՍԵՂՄԱԳԻՐ

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YEREVAN STATE UNIVERSITY

Avetik Pahlevanyan

On some problems of spectral theory

SYNOPSIS

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A.01.02 – "Differential Equations, Mathematical Physics"

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Ատենախոսության թեման հաստատվել է ԵՊՀ մաթեմատիկայի և մեխանիկայի ֆա–կուլտետի խորհրդի կողմից:

Գիտական ղեկավար՝ ֆիզ.-մաթ. գիտ. դոկտոր Տ.Ն. Հարությունյան

Պաշտոնական ընդդիմախոսներ՝ ֆիզ.-մաթ. գիտ. դոկտոր Հ.Գ. Ղազարյան, ֆիզ.-մաթ. գիտ. դոկտոր Վ.Վ. Ներսեսյան (Ֆրանսիա)

Առաջատար կազմակերպություն՝ Հայ-Ռուսական (Սլավոնական) Համալսարան

Պաշտպանությունը կայանալու է 2018թ. մայիսի 22–ին, ժ. 15:00–ին ԵՊՀ-ում (0025, ք. Երևան, Ալեք Մանուկյան 1) գործող ԲՈՀ-ի 050 "Մաթեմատիկա" մասնագիտական խորհրդի նիստում:

Ատենախոսությանը կարելի է ծանոթանալ ԵՊՀ-ի գրադարանում:

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The topic of the thesis approved at a meeting of academic council of the Faculty of Mathematics and Mechanics of the Yerevan State University.

Scientific supervisor: Doctor of phys.-math. sciences T.N. Harutyunyan

Official opponents: Doctor of phys.-math. sciences H.G. Ghazaryan, l'Habilitation à Diriger des Recherches V.V. Nersesyan (France)

Leading institution: Russian-Armenian (Slavonic) University

Thesis defense will take place on May 22, 2018 at 15:00, during the meeting of the Higher Attestation Commission specialized council 050 "Mathematics" at YSU (1 Alex Manoogian, Yeravan 0025, Armenia).

The thesis is available in the library of the Yerevan State University.

The synopsis was sent on April 20, 2018.

Scientific secretary of specialized council: T.N. Harutyunyan

General description of the work

Relevance of the topic. Let us denote by $L(q,\alpha,\beta)$ the following Sturm-Liouville boundary value problem

$$-y'' + q(x)\,y = \mu y \equiv \lambda^2 y,\ x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{0.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,\ \alpha \in (0,\pi], \tag{0.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0,\ \beta \in [0,\pi), \tag{0.3}$$

where the potential q is a real-valued, summable function on $[0,\pi]$ (we write $q \in L^1_{\mathbb{R}}[0,\pi]$).

By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator generated by the problem (0.1)–(0.3) in Hilbert space $L^2[0,\pi]$ (see [1, 2]). It is well-known, that the spectrum of $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues (see [1, 2, 3]), which we denote by $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\dots$, emphasizing the dependence on q, α and β. We assume that eigenvalues μ_n are enumerated in increasing order:

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \dots < \mu_n(q,\alpha,\beta) < \dots .$$

Let $\varphi(x,\mu,\alpha)$ and $\psi(x,\mu,\beta)$ be the solutions of (0.1), satisfying the initial conditions

$$\varphi(0,\mu,\alpha) = \sin\alpha,\ \varphi'(0,\mu,\alpha) = -\cos\alpha, \tag{0.4}$$

$$\psi(\pi,\mu,\beta) = \sin\beta,\ \psi'(\pi,\mu,\beta) = -\cos\beta. \tag{0.5}$$

It is easy to see that the functions $\varphi_n(x) := \varphi(x,\mu_n,\alpha)$ and $\psi_n(x) := \psi(x,\mu_n,\beta)$, $n = 0,1,2,\dots$, are the eigenfunctions corresponding to the eigenvalue μ_n. The squares of the L^2 norms of these eigenfunctions:

$$a_n = a_n(q,\alpha,\beta) = \int_0^\pi \varphi_n^2(x)\,dx, \quad b_n = b_n(q,\alpha,\beta) = \int_0^\pi \psi_n^2(x)\,dx, \tag{0.6}$$

are called norming constants.

The existence, countability and asymptotic formulae for the eigenvalues, asymptotic formulae for the norming constants, completeness of the eigenfunctions in $L^2[0,\pi]$, convergence of expansion for eigenfunctions for various classes of functions, which are the ingredients of the direct Sturm-Liouville problem, have been studied since the mid-19th and early 20th century (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]) and have received a fairly complete form at the end of 20th century thanks to the works of Titchmarsh, Levitan, Marchenko and others in those case when $q \in L^2_{\mathbb{R}}[0,\pi]$, $\sin\alpha \neq 0$, $\sin\beta \neq 0$ and $\alpha = \pi$, $\beta = 0$ (see [18, 19, 2, 20, 21, 22, 23, 3]).

Inverse Sturm-Liouville problems originates from one work of Ambarzumian (dated 1929), where the following uniqueness theorem was proved:

Theorem 1 *([24]) If the eigenvalues* $\mu_n\left(q, \frac{\pi}{2}, \frac{\pi}{2}\right) = n^2$, $n \geq 0$, *then* $q(x) = 0$ *a.e. on* $(0,\pi)$.

The first, who pointed out the importance of the result of Ambarzumian, was Swedish mathematician Göran Borg. He showed [25] that Ambarzumian's result is an exception from the "rule" - one spectrum does not determine a Sturm-Liouville operator. In the same paper, Borg showed that two spectra of a Sturm-Liouville operator (for different boundary conditions) determine it uniquely.

In 1950 Marchenko, using transformation operators (for arbitrary Sturm-Liouville equations constructed by Povzner [26]), proved that the so-called spectral function determines the operator uniquely (see [27, 22]). In the case of regular operator (defined on a finite interval for summable potential) the spectral function is defined by two sequences - eigenvalues and the norming constants.

Other types of uniqueness theorems were studied by Levinson ([28]), Hochstadt and Lieberman ([29]), Isaacson and Trubowitz ([30]), McLaughlin and Rundell ([31]), McLaughlin ([32]), Harutyunyan ([33]) and others.

In 1951 in their fundamental work [34] Gelfand and Levitan succeeded in giving an efficient algorithm for constructive solution of inverse Sturm-Liouville problem through the spectral function for twice differentiable potential. In the case of regular operator their result provides necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$, to be the spectrum and the norming constants of a problem $L(q,\alpha,\beta)$, $\alpha,\beta \in (0,\pi)$, respectively. Afterwards this result was generalized (for more general classes of potentials and for different types of given spectral data) by descendants of Gelfand and Levitan and by other mathematicians (see [19, 35, 36, 37] and the references therein).

Another interesting approach for studying inverse Sturm-Liouville problems was suggested by E. Trubowitz and his colleagues (see [30, 38, 39, 36]). This approach is connected with the description of all problems of the form (0.1)–(0.3) that have the same spectrum (also called isospectrality problem). One of the main ingredients here is the classical Darboux transformation [40, 41], which allows authors in [30, 38] to reduce the problem of the form (0.1)–(0.3) ($\sin\alpha \neq 0$, $\sin\beta \neq 0$) to the equation of the form (0.1) with the Dirichlet boundary conditions ($y(0) = 0$, $y(\pi) = 0$) (Dirichlet problem). The whole book [36] is devoted to the study of the Dirichlet problem. Some aspects of this approach were also studied for the case $\sin\alpha = 0$, $\sin\beta \neq 0$ ($\alpha = \pi$, $\beta \in (0,\pi)$) (see [39, 42, 43]).

In 1997, Jodeit and Levitan (see [44]) suggested to use Gelfand-Levitan equation (see [34]) and transformation operators (see [26]) to deal with isospectrality problem. They themselves solved the isospectrality problem (explicit formulas are given for parameters determining boundary conditions, implicit formula is given for potential) under assumptions $q' \in L^2_{\mathbb{R}}[0,\pi]$ and $\sin\alpha \neq 0$, $\sin\beta \neq 0$ ($\alpha, \beta \in (0,\pi)$), with some remarks for the other cases at the end of the paper [44].

The recent research, especially in the theory of inverse Sturm-Liouville problems, shows that, in addition to such classical problems as vibrations of strings (see [45, 46, 47, 48, 49, 50, 51]), Schrödinger equation in quantum mechanics (see [52, 53, 54]) and Korteweg-de Vries equation in the theory of nonlinear waves (see [55, 40, 56]), Sturm-Liouville problems arise in the theory of plasma dynamics (see [57]), in the biomedical engineering (see [58]) and in many other branches of applied sciences.

Objectives. Despite the extensive study in the field of the inverse Sturm-Liouville problems, to our knowledge, so far, the necessary and sufficient conditions for the sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be the spectrum and the norming constants for the problem $L(q,\pi,\beta)$ (analogously for $L(q,\alpha,0)$) with $q \in L^1_{\mathbb{R}}[0,\pi]$ have not been found, which in particularly means that the constructive solution of the inverse Sturm-Liouville problem in this case is not given. Our primary goal within the thesis is to find these conditions and provide an efficient algorithm for the constructive solution of the inverse problem. With this purpose, in the thesis the following problems, also having separate interest, are studied.

1) Uniform convergence of the expansion of an absolutely continuous function for eigenfunctions of the Sturm-Liouville problems $L(q,\pi,\beta)$, $\beta \in (0,\pi)$ and $L(q,\alpha,0)$, $\alpha \in (0,\pi)$, with summable potential $q \in L^1_{\mathbb{R}}[0,\pi]$;

2) Asymptotic formula for the eigenvalues of the problem $L(q,\pi,\beta)$, with $q \in L^1_{\mathbb{R}}[0,\pi]$ and $\beta \in (0,\pi)$;

3) Asymptotic formulae for the norming constants of the problem $L(q,\alpha,\beta)$,

with $q \in L^1_{\mathbb{R}}[0,\pi]$ and $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$;

4) Riesz basicity of the systems $\{\cos\sqrt{\mu_n}x\}_{n=0}^{\infty}$ and $\{\sin\sqrt{\mu_n}x\}_{n=0}^{\infty}$ in $L^2[0,\pi]$;

5) Derivation of an analogue of the Gelfand-Levitan equation for our case $\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0,\pi]$;

6) Existence and uniqueness of the solution of this Gelfand-Levitan equation as well as reconstruction of the function q (i.e. the reconstruction of differential equation) and the boundary conditions.

Besides, the dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters determining the boundary conditions is studied in the thesis.

Research methods. Methods of entire and meromorphic functions, methods of the theory of series, methods of differential and integral equations, as well as methods of functional analysis are used to obtain the results of the thesis.

Scientific novelty. All results of the thesis are new.

Theoretical and practical value. The results of the thesis have a theoretical value and can be applied in some direct and inverse Sturm-Liouville problems.

Approbation of the results. The main results of the thesis have been presented at the scientific seminars of the chair of "Differential Equations" at the Yerevan State University, at the scientific seminars of the department of "Differential and Integral Equations" and Young Scientists Council of the Institute of Mathematics of National Academy of Sciences of RA, at the international conferences "Harmonic Analysis and Approximations, V" (Tsaghkadzor, Armenia, 2011), "Mathematics in Armenia: Advances and Perspectives" (Tsaghkadzor, Armenia, 2013), "The IV International Conference of the Georgian Mathematical Union" (Tbilisi-Batumi, Georgia, 2013) and at the annual sessions of Armenian Mathematical Union (Yerevan, Armenia, 2012, 2013, 2015).

Publications. On the topic of the thesis 5 scientific articles and 6 abstracts of the conferences talks have been published, the list of which is provided at the end of the synopsis.

Structure and volume of the thesis. The thesis is exposed on 85 pages. It is comprised of Introduction, 3 Chapters divided into 9 Sections, Conclusion, Acknowledgments, Bibliography, consisting of 94 items, and the author's publications on the topic of the thesis (see [Art1–Art5] and [Abs1-Abs6]).

The main content of the thesis

In Section 1.1 of Chapter 1 of the thesis we investigate the uniform convergence of the expansion of an absolutely continuous function for eigenfunctions of the Sturm-Liouville problems. Namely, we prove the following two theorems:

Theorem 2 *Let $q \in L^1_{\mathbb{R}}[0,\pi]$, $\alpha = \pi$, $\beta \in (0,\pi)$ and f be an absolutely continuous function on $[0,\pi]$. Then for arbitrary $a \in (0,\pi)$*

$$\lim_{N\to\infty} \max_{x\in[a,\pi]} \left| f(x) - \sum_{n=0}^{N} c_n \varphi_n(x) \right| = 0, \; c_n = \frac{1}{a_n}\int_0^\pi f(t)\varphi_n(t)dt, \qquad (0.7)$$

where $\varphi_n(x) \equiv \varphi(x, \mu_n(q,\pi,\beta), \pi) \equiv \varphi(x, \mu_n, \pi)$.

Theorem 3 *Let $q \in L^1_{\mathbb{R}}[0,\pi]$, $\alpha \in (0,\pi)$, $\beta = 0$ and f be an absolutely continuous function on $[0,\pi]$. Then for arbitrary $b \in (0,\pi)$*

$$\lim_{N\to\infty} \max_{x\in[0,b]} \left| f(x) - \sum_{n=0}^{N} c_n \varphi_n(x) \right| = 0, \; c_n = \frac{1}{a_n}\int_0^\pi f(t)\varphi_n(t)dt, \qquad (0.8)$$

where $\varphi_n(x) \equiv \varphi(x, \mu_n(q,\alpha,0), \alpha)$.

These results are used to obtain new, more precise asymptotic formulae for the eigenvalues of the problem $L(q,\pi,\beta)$, with $q \in L^1_{\mathbb{R}}[0,\pi]$ and $\beta \in (0,\pi)$ and for the norming constants of the problem $L(q,\alpha,\beta)$, with $q \in L^1_{\mathbb{R}}[0,\pi]$ and $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$. Section 1.2 and Section 1.3 of Chapter 1 are devoted to the study of these problems. Before formulation of the results let us recall some concepts and results that will be used further.

In the paper [59] Harutyunyan, while studying the dependence of the eigenvalues on parameters α and β determining the boundary conditions (0.2) and (0.3), introduced the concept of the function of $\delta_n(\alpha,\beta)$, which is defined by the formula

$$\delta_n(\alpha,\beta) := \sqrt{\mu_n(0,\alpha,\beta)} - n = \lambda_n(0,\alpha,\beta) - \lambda_n\left(0,\frac{\pi}{2},\frac{\pi}{2}\right), \; n \geq 2, \qquad (0.9)$$

and proved that $-1 \leq \delta_n(\alpha,\beta) \leq 1$ and it is a solution of the following transcendental equation:

$$\delta = \frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{(n+\delta)^2\sin^2\alpha + \cos^2\alpha}} - \frac{1}{\pi}\arccos\frac{\cos\beta}{\sqrt{(n+\delta)^2\sin^2\beta + \cos^2\beta}}. \qquad (0.10)$$

In the thesis (see Section 2.4 of Chapter 2) we prove that the equation (0.10) has a unique solution for each fixed $n = n_0 \geq 2$, $\alpha = \alpha_0 \in (0, \pi]$, $\beta = \beta_0 \in [0, \pi)$.

The next theorem, proved by Harutyunyan in [60], gives more precise estimate of the remainder term in the asymptotic formulae for eigenvalues.

Theorem 4 *([60]) Let $q \in L^1_{\mathbb{R}}[0,\pi]$ and let $\lambda_n^2(q,\alpha,\beta) = \mu_n(q,\alpha,\beta)$. Then*

(a) The asymptotic relation $(n \to \infty)$

$$\lambda_n(q,\alpha,\beta) = n + \delta_n(\alpha,\beta) + \frac{[q]}{2(n+\delta_n(\alpha,\beta))} + l_n(q,\alpha,\beta) + O\left(\frac{1}{n^2}\right), \quad (0.11)$$

holds, where $[q] = \frac{1}{\pi}\int\limits_0^\pi q(t)\,dt$,

$$l_n(q,\alpha,\beta) = \frac{1}{2\pi(n+\delta_n(\alpha,\beta))}\int\limits_0^\pi q(x)\cos 2(n+\delta_n(\alpha,\beta))x dx, \ \alpha \in (0,\pi),$$

and

$$l_n(q,\pi,\beta) = -\frac{1}{2\pi(n+\delta_n(\pi,\beta))}\int\limits_0^\pi q(x)\cos 2(n+\delta_n(\pi,\beta))x dx. \quad (0.12)$$

The estimate $O\left(\frac{1}{n^2}\right)$ of the remainder in (0.11) is uniform in all $\alpha,\beta \in [0,\pi]$, and $q \in BL^1_{\mathbb{R}}[0,\pi]$ (here and below $BL^1_{\mathbb{R}}[0,\pi]$ stands for bounded subsets of $L^1_{\mathbb{R}}[0,\pi]$).

(b) For $\alpha,\beta \in (0,\pi)$ and for the case $\alpha = \pi$, $\beta = 0$ the function l, defined by the formula

$$l(x) = \sum_{n=2}^{\infty} l_n(q,\alpha,\beta)\sin(n+\delta_n(\alpha,\beta))x, \quad (0.13)$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$, that is $l \in AC(0,2\pi)$.

The proof of the Theorem 4 does not cover the case $\alpha = \pi$, $\beta \in (0,\pi)$. In Section 1.2 of Chapter 1 using Theorem 2 we handle this case and prove the following theorem:

Theorem 5 *The function l, defined by the formula*

$$l(x,\beta)=\sum_{n=2}^{\infty} l_n(q,\pi,\beta)\sin\left(n+\delta_n(\pi,\beta)\right)x$$

is absolutely continuous on arbitrary segment $[a,b]\subset(0,2\pi)$, i.e. $l\in AC(0,2\pi)$.

In Section 1.3 of Chapter 1 of the thesis the following two theorems are proved.

Theorem 6 *Let $q\in L^1_{\mathbb{R}}[0,\pi]$. For norming constants a_n and b_n the following asymptotic formulae hold (when $n\to\infty$):*

$$a_n(q,\alpha,\beta)=\frac{\pi}{2}\left[1+\frac{2\,s_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}+r_n\right]\sin^2\alpha+ \\ +\frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}\left[1+\frac{2\,s_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}+\tilde r_n\right]\cos^2\alpha, \quad (0.14)$$

$$b_n(q,\alpha,\beta)=\frac{\pi}{2}\left[1+\frac{2\,s_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}+p_n\right]\sin^2\beta+ \\ +\frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}\left[1+\frac{2\,s_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}+\tilde p_n\right]\cos^2\beta,$$

where

$$s_n=s_n(q,\alpha,\beta)=-\frac{1}{2}\int_0^{\pi}(\pi-t)\,q(t)\sin 2\left[n+\delta_n(\alpha,\beta)\right]t\,dt, \quad (0.15)$$

$r_n=r_n(q,\alpha,\beta)=O\left(\frac{1}{n^2}\right)$ *and* $\tilde r_n=\tilde r_n(q,\alpha,\beta)=O\left(\frac{1}{n^2}\right)$ *(the same estimate is true for p_n and $\tilde p_n$), when $n\to\infty$, uniformly in $\alpha,\beta\in[0,\pi]$ and $q\in BL^1_{\mathbb{R}}[0,\pi]$.*

Theorem 7 *The function s, defined by the formula*

$$s(x)=\sum_{n=2}^{\infty}\frac{s_n(q,\alpha,\beta)}{n+\delta_n(\alpha,\beta)}\cos\left(n+\delta_n(\alpha,\beta)\right)x \quad (0.16)$$

(a) is absolutely continuous on arbitrary segment $[a,b]\subset(0,2\pi)$, for both $\alpha,\beta\in(0,\pi)$ and $\alpha=\pi,\ \beta=0$;

(b) is absolutely continuous on $[0,2\pi]$, for $\alpha=\pi,\ \beta\in(0,\pi)$.

In Section 1.4 of Chapter 1 we investigate the Riesz basicity of the systems $\{\cos\lambda_n(q,\alpha,\beta)x\}_{n=0}^{\infty}$ and $\{\sin\lambda_n(q,\alpha,\beta)x\}_{n=0}^{\infty}$ in $L^2[0,\pi]$ and prove the following two theorems:

Theorem 8 *The system of functions* $\{\cos\lambda_n(q,\alpha,\beta)x\}_{n=0}^{\infty}$ *is a Riesz basis in* $L^2[0,\pi]$ *for each triple* $(q,\alpha,\beta)\in L^1_{\mathbb{R}}[0,\pi]\times(0,\pi]\times[0,\pi)$, *except one case: when* $\alpha=\pi$, $\beta=0$, *the system* $\{\cos\lambda_n(q,\pi,0)x\}_{n=0}^{\infty}$ *is not a basis, but the system* $\{\cos\lambda x\}\cup\{\cos\lambda_n(q,\pi,0)x\}_{n=0}^{\infty}$ *is a Riesz basis in* $L^2[0,\pi]$, *where* $\lambda^2\neq\lambda_n^2$ *for every* $n=0,1,2,\ldots$.

Theorem 9 *1. Let* $\alpha,\beta\in(0,\pi)$. *Then the systems*

(a) $\{\sin\lambda_n x\}_{n=1}^{\infty}$, *if there is no zeros among* $\lambda_n=\lambda_n(q,\alpha,\beta)$, $n=0,1,2,\ldots$ *(i.e. in this case we "throw away"* $\sin\lambda_0 x$*),*

(b) $\{\sin\lambda_n x\}_{n=0}^{n_0-1}\cup\{\sin\lambda_n x\}_{n=n_0+1}^{\infty}$, *if* $\lambda_{n_0}(q,\alpha,\beta)=0$ *(we "throw away"* $\sin\lambda_{n_0}x\equiv 0$*)*

are Riesz bases in $L^2[0,\pi]$.

2. Let $\alpha=\pi$, $\beta\in(0,\pi)$ *or* $\alpha\in(0,\pi)$, $\beta=0$. *Then the systems*

(a) $\{\sin\lambda_n x\}_{n=0}^{\infty}$, *if there is no zeros among* $\lambda_n=\lambda_n(q,\alpha,\beta)$, $n=0,1,2,\ldots$,

(b) $\{\sin\lambda_n x\}_{n=0}^{n_0-1}\cup\{x\}\cup\{\sin\lambda_n x\}_{n=n_0+1}^{\infty}$, *if* $\lambda_{n_0}=0$

are Riesz bases in $L^2[0,\pi]$.

3. Let $\alpha=\pi$, $\beta=0$. *The answer is the same as in case 2.*

In Section 2.1 of Chapter 2 of the thesis we derive an analogue of the Gelfand-Levitan equation for the case $q\in L^1_{\mathbb{R}}[0,\pi]$, $\alpha=\pi$, $\beta\in(0,\pi)$. Transformation (transmutation) operators play an important role here.

Theorem 10 *([37]) For the function* $\varphi(x,\mu,\pi)$ *the following representation holds*

$$\varphi(x,\mu,\pi)=(\mathbb{I}+\mathbb{P})\frac{\sin\lambda x}{\lambda}:=\frac{\sin\lambda x}{\lambda}+\int_0^x P(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{0.17}$$

where $P(x,t)$, $0\le t\le x\le\pi$, *is a real continuous function with the same smoothness as* $\int_0^x q(t)dt$, *and*

$$P(x,x)=\frac{1}{2}\int_0^x q(t)dt,\ P(x,0)=0. \tag{0.18}$$

The proof of the Theorem 10 was given for the case $q \in L^2_{\mathbb{R}}[0,\pi]$ (see [37]), but it can be easily done for the case $q \in L^1_{\mathbb{R}}[0,\pi]$, without any significant changes.

Let us now consider the function

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{1}{a_n} \frac{\sin \lambda_n x}{\lambda_n} \frac{\sin \lambda_n t}{\lambda_n} - \right.$$
$$\left. - \frac{1}{a_n(0,\pi,\beta)} \frac{\sin \lambda_n(0,\pi,\beta) x}{\lambda_n(0,\pi,\beta)} \frac{\sin \lambda_n(0,\pi,\beta) t}{\lambda_n(0,\pi,\beta)} \right), \qquad (0.19)$$

where $\{\lambda_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ are two sequences satisfying some asymptotic relations (see Theorem 12 below).

It is proved in the same Section 2.1, that $F(x,t)$ is a continuous function in $\{[0,\pi] \times [0,\pi]\} \setminus \{(\pi,\pi)\}$ and $\frac{d}{dx}F(x,x) \in L^1_{\mathbb{R}}(0,\pi)$. Further, the following theorem is proved:

Theorem 11 *For each fixed $x \in (0,\pi]$, the kernel $P(x,t)$ of the transformation operator (see (0.17)) satisfies the following linear integral equation*

$$P(x,t) + F(x,t) + \int_0^x P(x,s) F(s,t)\, ds = 0, \ 0 \le t < x, \qquad (0.20)$$

which is also called Gelfand-Levitan equation.

In Section 2.2 of Chapter 2 the following four lemmas, which provide the solution of the inverse Sturm-Liouville problem, are proved.

Lemma 1 *For each fixed $x \in (0,\pi]$, equation (0.20) has a unique solution $P(x,\cdot)$ in $L^2[0,x)$.*

Let us define

$$\varphi_\pi(x,\mu) \equiv \varphi_\pi(x,\lambda^2) := \frac{\sin \lambda x}{\lambda} + \int_0^x P(x,t) \frac{\sin \lambda t}{\lambda} dt, \quad q(x) := 2\frac{d}{dx}P(x,x).$$

Lemma 2 *The following relations hold*

$$-\varphi_\pi''(x,\mu) + q(x)\varphi_\pi(x,\mu) = \mu \varphi_\pi(x,\mu), \qquad (0.21)$$

$$\varphi_\pi(0,\mu) = 0, \ \varphi_\pi'(0,\mu) = 1. \qquad (0.22)$$

Lemma 3 *The following relation holds*

$$\int\limits_0^\pi \varphi_\pi(t,\mu_k)\,\varphi_\pi(t,\mu_n)\,dt = \begin{cases} 0, & n \neq k, \\ a_n, & n = k. \end{cases} \tag{0.23}$$

Lemma 4 *For all $n, m \geq 0$*

$$\frac{\varphi_\pi'(\pi,\mu_n)}{\varphi_\pi(\pi,\mu_n)} = \frac{\varphi_\pi'(\pi,\mu_m)}{\varphi_\pi(\pi,\mu_m)} = const. \tag{0.24}$$

Thus, the following result holds:

Theorem 12 *For two real sequences $\{\mu_n\}_{n=0}^\infty$ and $\{a_n\}_{n=0}^\infty$ to be the spectrum and the norming constants of a problem $L(q,\pi,\beta)$, with $q \in L^1_{\mathbb{R}}[0,\pi]$ and some $\beta \in (0,\pi)$, it is necessary and sufficient that the following relations hold:*

$$\sqrt{\mu_n} \equiv \lambda_n = n + \delta_n(\pi,\beta) + \frac{c}{2(n+\delta_n(\pi,\beta))} + l_n,\ \mu_n \neq \mu_m\ (n \neq m), \tag{0.25}$$

$$a_n = \frac{\pi}{2(n+\delta_n(\pi,\beta))^2}\left(1 + \frac{2\,s_n}{\pi\,(n+\delta_n(\pi,\beta))}\right),\ a_n > 0, \tag{0.26}$$

where c is a constant, the remainders $l_n = o\left(\frac{1}{n}\right)$ (when $n \to \infty$) are such that, the function

$$l(t) = \sum_{n=2}^\infty l_n \sin(n+\delta_n(\pi,\beta))\,t \tag{0.27}$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$ (we will write $l \in AC(0,2\pi)$) and the remainders $s_n = o(1)$ (when $n \to \infty$) are such that, the function

$$s(t) = \sum_{n=2}^\infty \frac{s_n}{n+\delta_n(\pi,\beta)} \cos(n+\delta_n(\pi,\beta))\,t \tag{0.28}$$

is absolutely continuous on $[0,2\pi]$ (we will write $s \in AC[0,2\pi]$).

In Section 3.1 of Chapter 3 we study the dependence of the zeros of eigenfunctions of Sturm-Liouville problem on the parameters that determining the boundary conditions. As a corollary, we obtain Sturm oscillation theorem (for $q \in L^1_{\mathbb{R}}[0,\pi]$).

Theorem 13 *Let $q \in L^1_{\mathbb{R}}[0,\pi]$. Then the eigenfunctions of the problem $L(q,\alpha,\beta)$ corresponding to the n-th eigenvalue $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots,$ have exactly n zeros in $(0,\pi)$. All these zeros are simple. If $\alpha = \pi$ and $\beta = 0$, then the n-th eigenfunction has $n+2$ zeros in $[0,\pi]$, and if either $\alpha = \pi$, $\beta \in (0,\pi)$ or $\beta = 0$, $\alpha \in (0,\pi)$, then the n-th eigenfunction has $n+1$ zeros in $[0,\pi]$.*

Bibliography

[1] M.A. Naimark. *Linear differential operators (in Russian).* Izdat. "Nauka", Moscow, 1969. Second edition, revised and augmented, With an appendix by V. È. Ljance.

[2] B.M. Levitan and I.S. Sargsyan. *Introduction to spectral theory. Selfadjoint ordinary differential operators (in Russian).* Izdat. "Nauka", Moscow, 1970.

[3] V.A. Marchenko. *Sturm-Liouville operators and their applications (in Russian).* Izdat. "Naukova Dumka", Kiev, 1977.

[4] Charles Sturm. Mémoire sur les équations différentielles linéaires du second ordre. In *Collected Works of Charles François Sturm*, pages 392–472. Springer, 2009.

[5] Charles Sturm. Mémoire sur une classe d'équations à différences partielles. In *Collected Works of Charles François Sturm*, pages 505–576. Springer, 2009.

[6] Joseph Liouville. Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable. *Journal de mathématiques pures et appliquées*, pages 253–265, 1836.

[7] Joseph Liouville. Démonstration d'un théorème dû à M. Sturm, et relatif à une classe de fonctions transcendantes. *Journal de mathématiques pures et appliquées*, pages 269–277, 1836.

[8] Charles Sturm and Joseph Liouville. Extrait d'un Mémoire sur le développement des fonctions en séries dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable. *Collected Works of Charles François Sturm*, pages 584–587, 2009.

[9] Joseph Liouville. Solution d'un Problème d'analyse. *Journal de mathématiques pures et appliquées*, pages 1–2, 1837.

[10] Joseph Liouville. Second Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujétis à satisfaire a une meme equation differentielle du second ordre, contenant un parametre variable. *Journal de mathématiques pures et appliquées*, pages 16–35, 1837.

[11] Joseph Liouville. Sur la sommation d'une série. *Journal de mathématiques pures et appliquées*, pages 107–108, 1837.

[12] V.A. Steklov. The problem of cooling an inhomogeneous solid rod. *Communications of the Kharkov Mathematical Society*, 5:136–181, 1897.

[13] V.A. Steklov. Sur le problème de refroidissement d'une barre hétérogène. *Comp. Rend. Acad. Sci. Paris*, 126:215–218, 1898.

[14] Adolf Kneser. Untersuchungen über die Darstellung willkürlicher Funktionen in der mathematischen Physik. *Mathematische Annalen*, 58(1):81–147, 1903.

[15] E.W. Hobson. On a general convergence theorem, and the theory of the representation of a function by series of normal functions. *Proceedings of the London Mathematical Society*, 2(1):349–395, 1908.

[16] George D. Birkhoff. Boundary value and expansion problems of ordinary linear differential equations. *Trans. Amer. Math. Soc.*, 9(4):373–395, 1908.

[17] Alfred Haar. Zur theorie der orthogonalen funktionensysteme. *Mathematische Annalen*, 69(3):331–371, 1910.

[18] E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations. Part I.* Second Edition. Clarendon Press, Oxford, 1962.

[19] B.M. Levitan and M.G. Gasymov. Determination of a differential equation by two spectra (in Russian). *Uspehi Mat. Nauk*, 19(2 (116)):3–63, 1964.

[20] B.M. Levitan. *Inverse Sturm-Liouville problems (in Russian).* "Nauka", Moscow, 1984.

[21] B.M. Levitan and I.S. Sargsyan. *Sturm-Liouville and Dirac operators (in Russian).* "Nauka", Moscow, 1988.

[22] V.A. Marchenko. Some questions of the theory of one-dimensional linear differential operators of the second order. I (in Russian). *Trudy Moskov. Mat. Obsh.*, 1:327–420, 1952.

[23] V.A. Marchenko. *Spectral theory of Sturm-Liouville operators (in Russian)*. Izdat. "Naukova Dumka", Kiev, 1972.

[24] V.A. Ambarzumian. Über eine frage der eigenwerttheorie. *Zeitschrift für Physik A Hadrons and Nuclei*, 53(9):690–695, 1929.

[25] Göran Borg. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math.*, 78:1–96, 1946.

[26] A. Povzner. On differential equations of Sturm-Liouville type on a half-axis (in Russian). *Mat. Sbornik N.S.*, 23(65):3–52, 1948.

[27] V.A. Marchenko. Concerning the theory of a differential operator of the second order (in Russian). *Doklady Akad. Nauk SSSR. (N.S.)*, 72:457–460, 1950.

[28] Norman Levinson. The inverse Sturm-Liouville problem. *Mat. Tidsskr. B.*, 1949:25–30, 1949.

[29] Harry Hochstadt and Burton Lieberman. An inverse Sturm-Liouville problem with mixed given data. *SIAM J. Appl. Math.*, 34(4):676–680, 1978.

[30] E.L. Isaacson and E. Trubowitz. The inverse Sturm-Liouville problem. I. *Comm. Pure Appl. Math.*, 36(6):767–783, 1983.

[31] Joyce R. McLaughlin and William Rundell. A uniqueness theorem for an inverse Sturm-Liouville problem. *J. Math. Phys.*, 28(7):1471–1472, 1987.

[32] Joyce R. McLaughlin. Inverse spectral theory using nodal points as data—a uniqueness result. *J. Differential Equations*, 73(2):354–362, 1988.

[33] T.N. Harutyunyan. On a uniqueness theorem in the inverse Sturm-Liouville problem. *Mat. Vesnik*, 61(2):139–147, 2009.

[34] I.M. Gelfand and B.M. Levitan. On the determination of a differential equation from its spectral function (in Russian). *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 15:309–360, 1951.

[35] V.V. Zhikov. Inverse Sturm-Liouville problems on a finite interval (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:965–976, 1967.

[36] Jürgen Pöschel and Eugene Trubowitz. *Inverse spectral theory*, volume 130 of *Pure and Applied Mathematics.* Academic Press, Inc., Boston, MA, 1987.

[37] G. Freiling and V. Yurko. *Inverse Sturm-Liouville problems and their applications.* Nova Science Publishers, Inc., Huntington, NY, 2001.

[38] E.L. Isaacson, H.P. McKean, and E. Trubowitz. The inverse Sturm-Liouville problem. II. *Comm. Pure Appl. Math.*, 37(1):1–11, 1984.

[39] Björn E.J. Dahlberg and Eugene Trubowitz. The inverse Sturm-Liouville problem. III. *Comm. Pure Appl. Math.*, 37(2):255–267, 1984.

[40] V.B. Matveev and M.A. Salle. *Darboux transformations and solitons.* Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1991.

[41] M.M. Crum. Associated Sturm-Liouville systems. *Quart. J. Math. Oxford Ser. (2)*, 6:121–127, 1955.

[42] T.N. Harutyunyan and V.V. Nersesyan. A uniqueness theorem in the inverse Sturm-Liouville problem (in Russian). *Izv. Nats. Akad. Nauk Armenii Mat.*, 39(6):29–38, 2004.

[43] E. Korotyaev and D. Chelkak. The inverse Sturm–Liouville problem with mixed boundary conditions. *St. Petersburg Mathematical Journal*, 21(5):761–778, 2010.

[44] Max Jodeit, Jr. and B.M. Levitan. The isospectrality problem for the classical Sturm-Liouville equation. *Adv. Differential Equations*, 2(2):297–318, 1997.

[45] Brook Taylor. De motu nervi tensi. *Philosophical Transactions of the Royal Society of London*, 28:26–32, 1713.

[46] Johann Bernoulli. Meditationes de chordis vibrantibus. *Comment. Acad. Sci. Imper. Petropol*, 3:13–28, 1728.

[47] Daniel Bernoulli. Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae. *Comm. Acad. Sci. Petrop*, 6(1732-1733):108–122, 1738.

[48] Daniel Bernoulli. Demonstrationes theorematum suorum de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae. *Acad. Sci. Imp. Petr*, 7:1734–1735, 1740.

[49] Jean le Rond d'Alembert. publ. 1749. Recherches sur la courbe que forme une corde tendue mise en vibration. *Hist. Acad. Roy. Sci. et Belles Lettres Berlin*, pages 214–219.

[50] Jean le Rond d'Alembert. publ. 1770. Extrait de différentes lettres de m. d'alembert à m. de la grange. *Hist. Acad. Roy. Sci. et Belles Lettres Berlin*, pages 235–277, 1763.

[51] Leonhard Euler. De motu vibratorio tympanorum. *Novi Commentari Acad Petropolit*, 10:243–260, 1766.

[52] Erwin Schrödinger. An undulatory theory of the mechanics of atoms and molecules. *Physical Review*, 28(6):1049–1070, 1926.

[53] K. Chadan and P.C. Sabatier. *Inverse problems in quantum scattering theory.* Springer-Verlag, New York, NY, 1977.

[54] Khosrow Chadan, David Colton, William Rundell, and Lassi Päivärinta. *An introduction to inverse scattering and inverse spectral problems*, volume 2. Siam, 1997.

[55] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Method for solving the Korteweg-deVries equation. *Physical Review Letters*, 19(19):1095–1097, 1967.

[56] Francesco Calogero and Antonio Degasperis. *Spectral Transform and Soliton: Tools to Solve and Investigate Nonlinear Evolution Equations.* North-Holland, 1982.

[57] Jon Wilkening and Antoine Cerfon. A spectral transform method for singular Sturm–Liouville problems with applications to energy diffusion in plasma physics. *SIAM Journal on Applied Mathematics*, 75(2):350–392, 2015.

[58] Kun Gou and Zi Chen. Inverse Sturm-Liouville problems and their biomedical engineering applications. *JSM Math Stat*, 2015.

[59] T.N. Harutyunyan. The dependence of the eigenvalues of the Sturm-Liouville problem on boundary conditions. *Mat. Vesnik*, 60(4):285–294, 2008.

[60] T.N. Harutyunyan. Asymptotics of the eigenvalues of the Sturm-Liouville problem (in Russian). *Izv. Nats. Akad. Nauk Armenii Mat.*, 51(4):3–16, 2016.

Author's publication on the topic of the thesis

[Art1] A.A. Pahlevanyan. Convergence of expansions for eigenfunctions and asymptotics of the spectral data of the Sturm-Liouville problem (in Russian). *Izv. Nats. Akad. Nauk Armenii Mat.*, 52(6):77–90, 2017.

[Art2] T. Harutyunyan and A. Pahlevanyan. On the norming constants of the Sturm-Liouville problem. *Vestnik Kazan. Gos. Energet. Univ.*, (3 (31)):7–26, 2016.

[Art3] T. Harutyunyan, A. Pahlevanyan and Yu. Ashrafyan. On the "movement" of the zeros of eigenfunctions of the Sturm-Liouville problem (in Russian). *Proceedings of Artsakh State University*, (1 (27)):12–20, 2013.

[Art4] Tigran Harutyunyan, Avetik Pahlevanyan and Anna Srapionyan. Riesz bases generated by the spectra of Sturm–Liouville problems. *Electron. J. Differential Equations*, 2013(71):1–8, 2013.

[Art5] T.N. Harutyunyan, A.A. Pahlevanyan and A.V. Srapionyan. Riesz bases borned by the spectra of Sturm-Liouville problem (in Russian). *Mathematics in Higher School*, 8(3):36–44, 2012.

[Abs1] A.A. Pahlevanyan. *On the constructive solution of an inverse Sturm-Liouville problem*, Armenian Mathematical Union Annual Session 2015 dedicated to 100th anniversary of professor Haik Badalyan, June 23–25, 2015, Yerevan, Armenia, pp. 74–75.

[Abs2] Avetik Pahlevanyan. *The description of the isospectral Sturm-Liouville operators*, IV International Conference of the Georgian Mathematical Union dedicated to academician Victor Kupradze on the occasion of 110th anniversary of his birthday, September 9–15, 2013, Tbilisi-Batumi, Georgia, p. 105.

[Abs3] A. Pahlevanyan. *The isospectral Sturm-Liouville problems*, Second International Conference "Mathematics in Armenia: Advances and Perspectives" dedicated to the 70th anniversary of foundation of Armenian National Academy of Sciences, August 24–31, 2013, Tsaghkadzor, Armenia, pp. 76–77.

[Abs4] T.N. Harutyunyan and A.A. Pahlevanyan. *The uniqueness of the solution of a Gelfand-Levitan equation*, Armenian Mathematical Union Annual Session 2013 dedicated to 90th anniversary of academician Rafael Alexandrian, May 29 – June 1, 2013, Yerevan, Armenia, p. 62.

[Abs5] T. Harutyunyan, A. Pahlevanyan and A. Srapionyan. *Riesz bases connected with the solutions of Sturm-Liouville equations*, Armenian Mathematical Union Annual Session 2012 dedicated to the 1400th anniversary of Anania Shirakatsi, May 27 – June 2, 2012, Yerevan, Armenia, pp. 59–60.

[Abs6] T. Harutyunyan and A. Pahlevanyan. *On the solution of an inverse Sturm-Liouville problem*, International Conference "Harmonic Analysis and Approximations, V" dedicated to 75th anniversary of academician Norair Arakelian, September 10–17, 2011, Tsaghkadzor, Armenia, pp. 49–50.

ԱՄՓՈՓՈՒՄ

Ատենախոսության հիմնական արդյունքները հետևյալն են՝

1) $\{\mu_n\}_{n=0}^{\infty}$ և $\{a_n\}_{n=0}^{\infty}$ հաջորդականությունների համար ստացված են անհրաժեշտ և բավարար պայմաններ, որոնց դեպքում նրանք հանդիսանում են $L(q, \pi, \beta)$, $q \in L^1_{\mathbb{R}}[0, \pi]$ և որևէ $\beta \in (0, \pi)$ խնդրի սեփական արժեքներ և նորմավորող հաստատուններ;

2) Ապացուցված է բացարձակ անընդհատ ֆունկցիայի ըստ Շտուրմ-Լիուվիլի $L(q, \pi, \beta)$, $\beta \in (0, \pi)$ և $L(q, \alpha, 0)$, $\alpha \in (0, \pi)$ խնդիրների սեփական ֆունկ–ցիաների վերլուծության հավասարաչափ զուգամիտությունը հանրագումարելի պոտենցիալի դեպքում;

3) Ստացված է $L(q, \pi, \beta)$, $q \in L^1_{\mathbb{R}}[0, \pi]$ և $\beta \in (0, \pi)$ խնդրի սեփական արժեքների ավելի ճշգրիտ ասիմպտոտական բանաձև;

4) Ստացված են $L(q, \alpha, \beta)$, $q \in L^1_{\mathbb{R}}[0, \pi]$ և $(\alpha, \beta) \in (0, \pi] \times [0, \pi)$ խնդրի նորմա–վորող հաստատունների նոր և ավելի ճշգրիտ ասիմպտոտական բանաձևեր;

5) Հետազոտված է $\{\cos \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ և $\{\sin \lambda_n(q, \alpha, \beta)x\}_{n=0}^{\infty}$ համակարգերի բազիսությունը ըստ Ռիսի $L^2[0, \pi]$ տարածությունում;

6) Ստացված է Գելֆանդ-Լևիտանի հավասարումը $\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0, \pi]$ դեպքում;

7) Ապացուցված է Գելֆանդ-Լևիտանի հավասարման $\left(\alpha = \pi,\ q \in L^1_{\mathbb{R}}[0, \pi]\right)$ լուծ–ման գոյությունը և միակությունը;

8) Ներկայացված է Շտուրմ-Լիուվիլի $L(q, \pi, \beta)$, $q \in L^1_{\mathbb{R}}[0, \pi]$, $\beta \in (0, \pi)$ խնդրի վերականգնման արդյունավետ ալգորիթմ;

9) Ուսումնասիրված է Շտուրմ-Լիուվիլի եզրային խնդրի սեփական ֆունկցիաների զրոների կախվածությունը եզրային պայմանները որոշող պարամետրերից: Որ–պես հետևանք ստացված է Շտուրմի օսցիլյացիայի թեորեմը ($q \in L^1_{\mathbb{R}}[0, \pi]$-ի համար):

Заключение

Основными результатами диссертации являются:

1) Получены необходимые и достаточные условия на последовательности $\{\mu_n\}_{n=0}^{\infty}$ и $\{a_n\}_{n=0}^{\infty}$, при которых они являются собственными значениями и нормировочными постоянными краевой задачи $L(q,\pi,\beta)$, $q \in L^1_{\mathbb{R}}[0,\pi]$ и некоторой $\beta \in (0,\pi)$;

2) Доказана равномерная сходимость разложения абсолютно непрерывной функции по собственным функциям задачи Штурма-Лиувилля $L(q,\pi,\beta)$, $\beta \in (0,\pi)$ и $L(q,\alpha,0)$, $\alpha \in (0,\pi)$ с суммируемым потенциалом $q \in L^1_{\mathbb{R}}[0,\pi]$;

3) Получена более точная асимптотическая формула для собственных значений задачи $L(q,\pi,\beta)$, $q \in L^1_{\mathbb{R}}[0,\pi]$, и $\beta \in (0,\pi)$;

4) Получены новые, более точные асимптотические формулы для нормировочных постоянных задачи $L(q,\alpha,\beta)$, $q \in L^1_{\mathbb{R}}[0,\pi]$ и $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$;

5) Изучена базисность по Риссу систем функций $\{\cos\lambda_n(q,\alpha,\beta)x\}_{n=0}^{\infty}$ и $\{\sin\lambda_n(q,\alpha,\beta)x\}_{n=0}^{\infty}$ в пространстве $L^2[0,\pi]$;

6) Получено уравнение Гельфанда-Левитана для случая $\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0,\pi]$;

7) Доказаны существование и единственность решения уравнения Гельфанда-Левитана в случае $\alpha = \pi$, $q \in L^1_{\mathbb{R}}[0,\pi]$;

8) Представлен эффективный алгоритм восстановления задачи Штурма-Лиувилля $L(q,\pi,\beta)$, $q \in L^1_{\mathbb{R}}[0,\pi]$, $\beta \in (0,\pi)$;

9) Изучена зависимость нулей собственных функций краевой задачи Штурма-Лиувилля от параметров, определяющих краевые условия. В качестве следствия получена теорема осцилляции Штурма (для случая $q \in L^1_{\mathbb{R}}[0,\pi]$).