ՀՀ ԿՐԹՈՒԹՅԱՆ ԵՎ ԳԻՏՈՒԹՅԱՆ ՆԱԽԱՐԱՐՈՒԹՅՈՒՆ
ՀԱՅԱՍՏԱՆԻ ԱԶԳԱՅԻՆ ՊՈԼԻՏԵԽՆԻԿԱԿԱՆ ՀԱՄԱԼՍԱՐԱՆ
(ՀԻՄՆԱԴՐԱՄ)

Մոհամմադ Հասսան Մոհամմադի

Եզրային խնդիրներ էլիպսական և պարաբոլական հավասարումների համար և կիրառություններ երկֆազ միջավայրի մաթեմատիկական մոդելներում

Ա.01.02 – «Դիֆերենցիալ հավասարումներ, մաթեմատիկական ֆիզիկա»
մասնագիտությամբ ֆիզիկամաթեմատիկական գիտությունների թեկնածուի
գիտական աստիճանի հայցման ատենախոսության

ՍԵՂՄԱԳԻՐ

Երևան – 2018

Ministry of Education and Science of the Republic of Armenia
National Polytechnic University of Armenia (Foundation)

Mohammad Hassan Mohammadi

Boundary Value Problems for Elliptic and Parabolic Equations and Applications in Mathematical Modeling of Two-Phase Substance

ABSTRACT

Of thesis for requesting the degree of candidate of
Physical and mathematical sciences specializing in
01.01.02 “Differential Equations, mathematical physics”

Yerevan – 2018

Ատենախոսության թեման հաստատվել է ՀՀ ԳԱԱ Մաթեմատիկայի ինստիտուտի գիտական խորհուրդի նիստում

Գիտական ղեկավար՝	ֆ.մ.գ.դ., պրոֆ. *Ա.Հ.Բաբայան*
Պաշտոնական ընդդիմախոսներ՝	ֆ.մ.գ.դ., պրոֆ. *Ա.Խ.Խաչատրյան* ֆ.մ.գ.թ., դոցենտ. Ա.Հ.*Քամալյան*
Առաջատար կազմակերպություն՝	Խ. Աբովյանի անվան Հայկական պետական մանկավարժական համալսարան

Պաշտպանությունը կայանալու է 2018թ. դեկտեմբերի 25-ին, ժամը 15:00-ին Հայաստանի ազգային պոլիտեխնիկական համալսարանում գործող 053 մասնագիտական խորհրդի նիստում:

Հասցե՝ ք. Երևան 0009, փ. Տերյան 105, 12 մասնաշենք:

Ատենախոսությանը կարելի է ծանոթանալ ՀԱՊՀ գրադարանում:

Սեղմագիրն առաքված է 2018թ. նոյեմբերի 8-ին:

053 մասնագիտական խորհրդի
գիտ. քարտուղար,ֆ.մ.գ.դ., պրոֆ. Ա.Հ.Բաբայան

The title of thesis was approved at the meeting of a scientific council of Institute of Mathematics of National Academy of Sciences of Armenia

Scientific supervisor:	Doctor of Sciences, Prof. *A.H.Babayan*
Official opponents:	Doctor of Sciences, Prof. *A.Kh. Khachatryan* Ph.D., Docent. *A.H.Qamalyan*
Leading organization:	Armenian State Pedagogical University after Kh. Abovyan

The presentation will take place on the 25th of December 2018 at 15:00 at a meeting of 053 Mathematics specialized council acting in National Polytechnic University of Armenia.

Address: Yerevan 0009, 105 Teryan st., 12 bld.

The thesis can be found in the library of NPUA.

Abstract was sent out on 20th of November 2018.

Academic secretary of
053 specialized council, Doctor of Sciences, Prof. A.H.Babayan

General characterization of the thesis

The thesis is a research about free boundary problems, and more exactly it investigates the mathematical modeling of glass melting process and heat transfer phenomena according to classical rules in physics. Indeed, the main purpose of the present work is the construction of an effective numerical approach for the solution of free boundary problem which is based on finite element method. The boundary value problems for elliptic equations are considered also.

Actuality of the Subject

Mathematical modeling of heat transfer (Transport Phenomena) is applied by researchers to investigate the environment of the furnaces during the melting process and turbulent conditions inside them, because it has the advantages of reasonable cost and applicable exactness. Different researchers were worked about the free boundary problems and their mathematical characteristics, and Conservation laws were applied by physicists to construct the mathematical formulation of melting and freezing process for different types of materials. Stefan problem is famous between researches that it describes the behavior of ice block and its temperature distribution during the phase transition process.

Shoshana Kamin in 1958 prepared the first proof of the existence and uniqueness of the generalized solution of the Stefan problem in three-dimension. Her work was followed by Oleink, O. A. and she generalized the proof in the work "A method of solution of the general Stefan problem" in 1960. Meirmanov prepared comprehensive mathematical information about the Stefan problem in the text book "The Stefan Problem" in 1992, and also for existence and uniqueness of solution of Stefan problem and its regularity it is recommended to refer to Daniele Andreucci's work.

Obstacle problem is another important problem in the free and moving boundary problems topic. The aim of this problem is to look for the equilibrium of an elastic membrane whose boundary is constrained to hold above a given obstacle. Obstacle problem is applied in the fluid filtration, constrained heating, optimal control, financial mathematics and some other scopes. Caffarelli in the work "The obstacle problem revisited" reviewed some basic properties of obstacle problem and also Caffarelli and Riviere in the paper "Smoothness and analyticity of free boundaries in variational inequalities" in 1976 investigated more details in this scope. In the area of heat transfer and its mathematical simulation in three-dimensional case Raymond Viskanta and Aydin Ungan in 1986 prepared a paper which they presented a numerical approach to derive the modeling of circulation and heat transfer in an electrically boosted glass tank.

The finite element method (FEM) is a numerical approach for solving problems of mathematical physics. Typical problem areas of interest include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. The analytical solution of these problems generally require the solution to boundary value problems for partial differential equations. The method needs the variational formulation of the equations, then to solve the problem, it subdivides a large problem into smaller, simpler parts that are called finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function.

The earliest mathematical works on FEM be developed by Schellback in 1851 and Courant in1943. Also FEM was independently continued by engineers to address structural mechanics problems related to aerospace and civil engineering. The developments began in the mid-1950s with the papers of Turner, Clough, Martin and Topp in 1956, Argyris in 1957, and Babuska and Aziz in 1972. The books by Zienkiewicz in 1971 and Strang, and Fix in 1973 also laid the foundations for future development in FEM.

The Aim of the Thesis

1) To formulate the mathematical modeling of glass melting process in Garnissage furnace in three-dimension based on classical rules in physics (mass, momentum, and energy conservation laws) within Stefan condition.

2) To develop an algorithm, which reduces the free boundary problem (the nonlinear system of partial differential equations) to the new system of equations that has separated linear and nonlinear parts.
3) To investigate the numerical solution of the system according to finite element method in three-dimension.
4) To investigate the Dirichlet problem for fourth order partial differential equation in the unit disc. The equation is supposed elliptic and we consider properly and improperly elliptic equations.

Object of investigation

Free boundary problems within Stefan condition to prepare the mathematical modeling of melting process in the Garnissage furnace in three-dimension. Boundary value problems for elliptic equations

Methods of investigation

The methods include mathematical analysis, calculus of variations, linear algebra, matrix analysis, theory of free boundary problems, and the theory of finite element method for the nonlinear system of equations.

Scientific innovation

The thesis continues the works of Viskanta, Patankar, Zhao, Pilon, Rodrigues, Urbano, and Choudhary and according to their mathematical modeling we earn the modeling in three-dimension and by applying the calculus of variations we complete the process by performing the finite element method. The results about boundary value problem for elliptic equation are new also, and continued the investigations of N.E. Tovmasyan and A.H. Babayan.

Theoretical and Practical Value

The results obtained in the thesis have theoretical content and at the same time are directed toward applications. They can be used in the development of mathematical modeling of transport Phenomena and heat transfer.

Publications

The results of the thesis were published in 6 scientific articles which two journals indexed in SCOPUS.

Structure and Volume of the Thesis

The thesis consists of the introduction, four chapters, conclusion and the list of references. The number of references is 100, and the volume of the work is 115 pages.

The main content of the thesis

In introduction the actuality of the topics is discussed, some historical remarks are mentioned and the aim of investigation is explained.

Chapter 1. Some known definitions and necessary facts for following considerations are presented.

Section 1.1. Some historical notes and developing of free boundary problems and vital works in this domain were stated.

Section 1.2. We recall about basics of mathematical modeling.

Section 1.3. We state the basic definitions in physics which our modeling is related and constructed according to them.

Section 1.4. The mathematical modeling of continuity equation is prepared. To earn the model we applied the mass conservation law in classic physics and the result is partial differential equation in three-dimension.

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{V}) = 0$$

Section 1.5. The mathematical modeling of momentum equation is expressed. To gain the model we invoked the momentum conservation law in classic physics and the result is the system of partial differential equations in three-dimension.

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \boldsymbol{V}.\nabla(\rho \boldsymbol{V}) = -\nabla p + \nabla.\boldsymbol{\tau} + \rho \boldsymbol{g}$$

Section 1.6. The mathematical modeling of energy conservation equation is prepared. To obtain the model we used the energy conservation law in classic physics and the result is partial differential equation in three-dimension.

$$\rho c\boldsymbol{V}.\nabla\theta = \nabla.\left(k_{eff}\nabla\theta\right) + (\boldsymbol{\tau}:\nabla\boldsymbol{V}) + S$$

Section 1.7. The one-dimensional mathematical modeling of Stefan condition is obtained and the three-dimensional version of Stefan condition is concluded to perform in conduction equation.

$$\rho L h'(t) = -k_L\theta_x\,(h(t),t) + k_S\theta_x\,(h(t),t)$$

$$[\nabla\theta]_-^+.\boldsymbol{n}_x = -\lambda\boldsymbol{w}.\boldsymbol{n}_x = \lambda n_t$$

Section 1.8. We summarize the heat transfer equations and boundary conditions.

Transport phenomena system

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\boldsymbol{V}) = 0$$

$$\rho\left(\frac{\partial\boldsymbol{V}}{\partial t} + \boldsymbol{V}.\nabla\boldsymbol{V}\right) = -\nabla p + \nabla.\boldsymbol{\tau} + \rho\boldsymbol{g}$$

$$\rho c\boldsymbol{V}.\nabla\theta = k_{eff}\Delta\theta + (\boldsymbol{\tau}:\nabla\boldsymbol{V}) + S$$

Stefan condition

$$[\nabla\theta]_-^+.\boldsymbol{n}_x = -\lambda\boldsymbol{w}.\boldsymbol{n}_x = \lambda n_t \quad \text{on } \Gamma,$$

$$\Omega_1 = \{\theta > T_1\}, \qquad \Omega_2 = \{\theta < T_1\}, \qquad \Gamma = \{\theta = T_1\},$$

Other boundary and initial conditions

$$\theta|_{\Gamma} = T_1$$

$$\theta|_{t=0} = \psi(x),\ x \in \Omega_1$$

$$\theta|_{t=0} = \varphi(x),\ x \in \Omega_2$$

$$\boldsymbol{V}|_{\Gamma\cup\Omega_2} = 0$$

Section 1.9. Some basics and definitions of Sobolev spaces is prepared, also necessary norms and theorems about the convenient spaces for the solutions of partial differential equations is introduced.

$$H^1(\Omega) = \left\{ u \in L_2(\Omega) : \frac{\partial u}{\partial x_j} \in L_2(\Omega), j = 1,2, \dots, n \right\}$$

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L_2(\Omega)} + \sum_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}$$

Chapter 2. We derived the mathematical modelling of heat transfer in Garnissage furnace in two-dimensional case based on stream functions, and to illustrate the moving boundary between the solid and liquid phase we exert the Stefan condition, then we invoked the finite element method to gain the numerical solution of the system.

Section 2.1. Mathematical definitions of stream functions are stated.

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Section 2.2. We obtained the continuity equation based on stream functions and we have shown the stream functions satisfy in the continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Section 2.3. We earned the Navier-Stokes equations respect to stream functions and we gained two differential equations in two-dimension, then by subtraction we got the system as

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\Delta\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\Delta\psi) = \vartheta \Delta^2 \psi$$

Section 2.4. We derived the conduction equation according to stream functions.

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{k}{\rho c}\Delta\theta + \frac{1}{\rho c}S$$

Section 2.5. Weak formulation of Navier-Stokes and conduction equations were computed. This work has been done by applying sufficient smooth test functions, and also by inserting the Stefan condition into the energy equation, we obtained the variational version of modeling as

$$\iint_{\Omega} \Delta\psi \left(\frac{\partial\eta}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\eta}{\partial y}\frac{\partial\psi}{\partial x} + \vartheta\Delta\eta\right) dxdy = 0,$$

$$-\iint_{\Omega} \psi \left(\frac{\partial\theta}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\eta}{\partial x}\right) dxdy = -\frac{k}{\rho c}\iint_{\Omega} \nabla\theta.\nabla\eta dxdy +$$

$$\frac{k\lambda}{\rho c}\iint_{\Omega} \frac{\partial\eta}{\partial t} dxdy + \frac{S}{\rho c}\iint_{\Omega} \eta dxdy.$$

Section 2.6. To complete the process of the finite element approach we discrete the domain, and we replaced the approximate values ψ_h and θ_h instead of ψ and θ in the equalities, and we redefined the heat transfer problem as the problem of finding $\psi_h, \theta_h \in V_h$ such that ψ_h, θ_h satisfy in the relative integral equations. Then we got

$$\sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_i U_j \, a_{ijk} + \sum_{i=1}^{N(h)} U_i \, b_{ik} = 0$$

$$\sum_{i=1}^{N(h)} c_{ij} V_i = d_j \,;\; j = 1,2,3, \ldots, N(h).$$

Newton's method for nonlinear systems was recommended for the first nonlinear system, but the details of numerical approach is prepared in the chapter 4.

Chapter 3. We prepared some basic relations in vector analysis to apply them in the modeling process in cylindrical coordinate system to use convenient symmetric properties of furnace in the mathematical modeling process. We started the modeling in cylindrical coordinate and we derived the mathematical modeling and also its weak formulation, and finally we discrete the domain to replace the approximate variables. We completed the finite element method to obtain the numerical solution of the transport system. We have stated the conservation equations in the cylindrical coordinate, so the Stefan condition, then we converted the system of equations into the weak formulation and by applying the convenient test functions we discrete the domain to follow the finite element method. The Newton's method has been applied to derive the numerical solution of the system.

Section 3.1. Fundamentals and some basic definitions of vector analysis, like gradient and divergence in cylindrical coordinates are prepared.

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r}\frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} e_z,$$

$$\nabla . \boldsymbol{V} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}.$$

Section 3.2. Mathematical modeling of transport phenomena respect to cylindrical coordinate system has been computed. This essential part made according to the velocity vector field components u_r, u_φ.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial(\rho u_z)}{\partial z} = 0,$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right) + \rho g_r,$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z,$$

$$\frac{\partial \theta}{\partial t} + u_r\frac{\partial \theta}{\partial r} + u_z\frac{\partial \theta}{\partial z} = \frac{k}{\rho c}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{\partial^2 \theta}{\partial z^2}\right) + 2\vartheta|\boldsymbol{\tau}|^2 + \frac{1}{\rho c}F.$$

Section 3.3. Finite element method has been chosen to search the numerical solution of transport phenomena system, then we derived the variational version of system, then we will applied the convenient smooth test function $\eta(r, \varphi, z, t)$ with small support to earn the weak formulation.

$$\int_\Omega u_r\left(\frac{\eta}{r} - \frac{\partial \eta}{\partial r}\right) - \int_\Omega u_z\frac{\partial \eta}{\partial z} = 0,$$

$$\int_\Omega u_r\left(\frac{\partial \eta}{\partial t} + \left(\frac{1}{2}u_r + \frac{1}{r}\right)\frac{\partial \eta}{\partial r} + u_z\frac{\partial \eta}{\partial z} + \eta\frac{\partial u_z}{\partial z} - \frac{\partial^2 \eta}{\partial r^2} - \frac{\partial^2 \eta}{\partial z^2}\right) = -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial r} - \int_\Omega g_r\eta,$$

$$\int_\Omega u_z\left(\frac{\partial \eta}{\partial t} + u_r\frac{\partial \eta}{\partial r} + \eta\frac{\partial u_r}{\partial r} + \frac{1}{2}u_z\frac{\partial \eta}{\partial z} + \frac{\partial^2 \eta}{\partial r \partial z} + \frac{\partial^2 \eta}{\partial z^2}\right) = -\frac{1}{\rho}\int_\Omega p\frac{\partial \eta}{\partial z} - \int_\Omega g_z\eta,$$

$$\int_G \left(\theta + \frac{k}{\rho c}H(\theta)\right)\frac{\partial \eta}{\partial t} + \int_G \theta\left(u_r\frac{\partial \eta}{\partial r} + u_z\frac{\partial \eta}{\partial z} + \frac{k}{\rho c}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2}\right)\right)$$

$$= -\int_G \left(\frac{1}{\rho c}F + 2\vartheta|\boldsymbol{\tau}|^2\right)\eta.$$

Section 3.4. After discretization of the domain we redefined the heat transfer problem as the problem of finding u_{r_h}, u_{z_h}, p_h, and θ_h such that there satisfy in the continuity equation, Navier-Stokes equations, and heat conduction equation, then we got the system of equations

$$\sum_{i=1}^{N(h)} \left(U_{r_i} a_{ik} - U_{z_i} b_{ik}\right) = 0,$$

$$\sum_{i=1}^{N(h)} U_{r_i} A_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{r_j} B_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{r_i} U_{z_j} C_{ijk} + \sum_{i=1}^{N(h)} P_i D_{ik} + E_k = 0,$$

$$\sum_{i=1}^{N(h)} U_{z_i} A'_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{z_j} B'_{ijk} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} U_{z_i} U_{r_j} C'_{ijk} + \sum_{i=1}^{N(h)} P_i D'_{ik} + E'_k = 0,$$

$$\sum_{i=1}^{N(h)} \theta_i F_{ik} + \sum_{i=1}^{N(h)} \sum_{j=1}^{N(h)} \theta_i \left(U_{r_j} + U_{z_j}\right) G_{jk} + H_k = 0,$$

$$k = 1,2,3, \dots, N(h).$$

Section 3.5. Some description about the process of the work, finite element approach, and Newton's method was prepared.

Section 3.6. Mathematical modeling of transport phenomena based on cylindrical coordinate and the result system of equations and their relative coefficients was reviewed.

Section 3.7. We continued the process by computing the coefficients in the transport system, for this objective we divided the domain Ω to the mesh cubes, and we constructed the test functions on ijk −mesh cube. To define the test functions we divided the ijk −mesh cube into the 24 tetrahedrons.

Parts 1, 2 $\phi(x,y,t) = \frac{1}{h}(-x + x_i - t + t_k) + 1$	Parts 3, 4 $\phi(x,y,t) = \frac{1}{h}(-x + x_i - y + y_j) + 1$
Parts 5, 6 $\phi(x,y,t) = \frac{1}{h}(-y + y_j + t - t_k) + 1$	Parts 7, 8 $\phi(x,y,t) = \frac{1}{h}(x - x_i + t - t_k) + 1$
Parts 9, 10 $\phi(x,y,t) = \frac{1}{h}(x - x_i + y - y_j) + 1$	Parts 11, 12 $\phi(x,y,t) = \frac{1}{h}(y - y_j - t + t_k) + 1$
Parts 13, 14 $\phi(x,y,t) = \frac{1}{h}(-t + t_k) + 1$	Parts 15, 16 $\phi(x,y,t) = \frac{1}{h}(x - x_i) + 1$
Parts 17, 18 $\phi(x,y,t) = \frac{1}{h}(-y + y_j) + 1$	Parts 19, 20 $\phi(x,y,t) = \frac{1}{h}(t - t_k) + 1$

Parts 21, 22	Parts 23, 24
$\phi(x,y,t) = \frac{1}{h}(y - y_j) + 1$	$\phi(x,y,t) = \frac{1}{h}(-x + x_i) + 1$

Section 3.8. We computed the coefficients of the continuity equation a_{ik} and b_{ik}, then we continued the process by determining the a_{ik} for $k = i, k = i - (N^2 + N - 1)$, and $k = i + (N^2 + N - 1)$.

$$a_{ii} = \frac{(x_i + h)(6x_i^2 - 8x_i(x_i + h)^2 + 3(x_i + h)^3)}{6h^2} \ln\left(1 + \frac{h}{x_i}\right) -$$

$$\frac{x_i^4 - 4x_i h^3 + 3h^4}{6h^2} \ln\left(1 - \frac{h}{x_i}\right) - \frac{x_i(3x_i^2 + h^2)}{9h}$$

We expressed the continuity equation as

$$(P, Q)\begin{pmatrix} U_r \\ U_z \end{pmatrix} = 0,$$

and we derived the matrices P and Q respectively as

$$\begin{pmatrix}
0.41 & 0 & \dots & 0 & 0.0009 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.62 & 0 & \dots & 0 & 0.0001 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0.67 & 0 & \dots & 0 & -0.0003 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & -0.0006 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0.003 & 0 & \vdots & 0 & 0.41 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0.002 & 0 & \vdots & 0 & 0.62 & 0 & \dots & \ddots & 0.0009 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.002 & 0 & \vdots & 0 & 0.67 & 0 & \ddots & 0 & 0.0001 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.002 & 0 & \vdots & 0 & 0.41 & \ddots & \vdots & 0 & -0.0003 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.003 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & -0.0006 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \vdots & \ddots & 0.41 & 0 & \vdots & 0 & 0.0009 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0 & \ddots & 0 & 0.41 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & \ddots & 0 & 0 & 0.62 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0.003 & 0 & 0 & 0 & 0.41
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0.002 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0.002 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & \dots & 0 & 0.002 & \ddots & 0 & 0 & 0 & 0 & 0 \\
-0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \dots & \ddots & 0.002 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & 0 & \ddots & 0 & 0.002 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & 0 & \ddots & \vdots & 0 & 0.002 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & 0 & \ddots & 0 & \vdots & 0 & 0.002 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \vdots & \ddots & 0 & 0 & \vdots & 0 & 0.002 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.002 & 0 & \ddots & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.002 & \ddots & 0 & 0 & 0 & 0 & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -0.002 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Section 3.9. We achieved the coefficients of the Navier-Stokes system. We started by determining the coefficient A_{ik}, and as we done in other parts we suppose the ijk −mesh cube that it was divided to 24 tetrahedrons, then after integration process in the whole parts we gain

$$A_{ii} = -\frac{x_i^3 + h^3}{3h^2} \ln\left(1 + \frac{h}{x_i}\right) + \frac{x_i^3 - h^3}{3h^2} \ln\left(1 - \frac{h}{x_i}\right) + \frac{6x_i^2 + 2h^2}{9h},$$

$$A_{i,i-(N^2+N-1)} = -\frac{1}{18h}(6x_i^2 + 15x_i h + 11h^2) + \frac{(x_i + h)^3}{3h^2}\ln\left(1 + \frac{h}{x_i}\right) - \frac{h^2}{12},$$

$$A_{i,i+(N^2+N-1)} = -\frac{1}{18h}(6x_i^2 - 15x_i h + 11h^2) + \frac{(x_i - h)^3}{3h^2}\ln\left(1 - \frac{h}{x_i}\right) + \frac{h^2}{12}.$$

Then we have done the same process to determine the other coefficients in the system of Navier-Stokes equations. At the end of this section we got the nonlinear system of equations in 128 equations in 128 unknowns.

Section 3.10. We have shown in the continuity equation

$$U_r = -P^{-1}QU_z$$

Also from section (4.5) the Navier-Stokes system has the style

$$AU_r + \psi_1(U_r, U_z) + Dp + E = 0,$$

$$A'U_z + \psi_2(U_r, U_z) + D'p + E' = 0,$$

where the matrices A, D, A', and D' are determined. Since $D = -D'$, and

$$E = h^3 g_r \mathbf{1},$$

$$E' = h^3 g_z \mathbf{1}.$$

After summation of the Navier-Stokes equalities we got

$$AU_r + A'U_z + \psi(U_r, U_z) + E + E' = 0$$

We replaced the value of U_r into the last equation to get

$$(A' - AP^{-1}Q)U_z + \psi(U_z) = -h^3(g_r + g_z)\mathbf{1}$$

The concluded system has 64 equations within 64 variables. We derived its numerical solution by invoking the Newton's method. All numerical results and their relative graphs was produced.

Chapter 4. We investigate the Dirichlet boundary value problem for fourth order elliptic equations. We suppose, that the characteristic equation has one double-multiple root and two simple roots.

Section 4.1. We consider the Dirichlet problem for the fourth order properly elliptic equation with constant coefficients in the unit disc. Since the characteristic equation has one double root in upper half-plane and two different roots in the lower half-plane the solution must be found in the class of functions Hölder continuous with first order derivatives up to the boundary, then we obtained the new formula for the determination of the defect numbers.

Section 4.2. We assumed the unit disk $D = \{(x, y): x^2 + y^2 < 1\}$ in the complex plane, and we considered the fourth order elliptic differential equation in the domain D

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x, y) = 0, \ \ (x, y) \in D,$$

where A_k are the complex constants ($A_0 \neq 0$). We assumed that the last equation is properly elliptic. We searched the solution of the equation in the class of $C^4(D) \cap C^{(1,\alpha)}(\overline{D})$, which satisfies the Dirichlet conditions

$$u\Big|_{\Gamma} = f(x, y), \ \ \frac{\partial u}{\partial N}\Big|_{\Gamma} = g(x, y), \ \ (x, y) \in \Gamma,$$

where Γ is the boundary of D, also $f \in C^{(1,\alpha)}(\Gamma)$ and $g \in C^{(\alpha)}(\Gamma)$ are the given functions, and $\frac{\partial}{\partial N} = -\frac{\partial}{\partial r}$ is a differentiation in the inner normal direction to the boundary Γ, and

$$z = x + iy = re^{i\varphi}$$

We tried to determine the number of solvability conditions of the problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x, y) = 0, \ \ (x, y) \in D,$$

$$u\Big|_{\Gamma} = f(x, y), \ \ \frac{\partial u}{\partial N}\Big|_{\Gamma} = g(x, y), \ \ (x, y) \in \Gamma,$$

and the number of the linearly independent solutions of the homogeneous problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x, y) = 0, \ \ (x, y) \in D$$

$$u\Big|_{\Gamma} = 0, \ \ \frac{\partial u}{\partial N}\Big|_{\Gamma} = 0, \ \ (x, y) \in \Gamma,$$

that is the defect numbers of the problem. Let λ_j are the roots of the characteristic equation $\sum_{k=0}^{4} A_k \lambda^{4-k} = 0$. We have

$$\lambda_1 = \lambda_2, \ \ \Im\lambda_1 > 0,$$
$$\lambda_3 \neq \lambda_4, \ \ \Im\lambda_{l+2} < 0, \ \ l = 1,2,$$

Denoting $\mu = \frac{i-\lambda_1}{i+\lambda_1}$, $\nu_j = \frac{i+\lambda_{2+j}}{i-\lambda_{2+j}}$, $j = 1{,}2$, the obtained result may be formulated as follows

Theorem 4.1. Assume that $\sigma = \mu\nu_1$ and $\tau = \mu\nu_2$, then the problem

$$\sum_{k=0}^{4} A_k \frac{\partial^4 u}{\partial x^k \partial y^{4-k}} (x,y) = 0, \ \ (x,y) \in D,$$

$$u\Big|_{\Gamma} = f(x,y), \ \ \frac{\partial u}{\partial N}\Big|_{\Gamma} = g(x,y), \ \ (x,y) \in \Gamma,$$

is uniquely solvable if and only if the conditions

$$P_k(\sigma,\tau) = \sum_{m=0}^{k-1}\sum_{p=0}^{m}(m-p)(\sigma\tau)^p \sum_{j=0}^{m-p-1} \sigma^j \tau^{m-p-j} \neq 0, \tag{4.2.9}$$

for $k = 3,4, \dots$ hold. If the conditions (4.2.9) fail, that is $P_{k_0}(\sigma,\tau) = 0$ for some value $k_0 > 2$, then the homogeneous problem has one linearly independent solution which is polynomial of order $k_0 + 1$. The corresponding inhomogeneous problem has a solution if the boundary functions satisfy one linearly independent orthogonality condition. Therefore, the defect numbers of the problem are equal to the numbers k_0 for which $P_{k_0}(\sigma,\tau) = 0$.

Section 4.3. We investigated more about the conditions (4.2.9). First we represent the equation in the form

$$\left(P_3(\sigma,\tau)\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} + E\frac{\partial^4}{\partial z \partial \bar{z}^3} + H\frac{\partial^4}{\partial z^3 \partial \bar{z}} + \mu^2\frac{\partial^4}{\partial z^4} + \nu_1\nu_2\frac{\partial^4}{\partial \bar{z}^4}\right)u = 0,$$

where $P_3(\sigma,\tau)$ defined in (5.2.9). For $k = 3$

$$P_3(\sigma,\tau) = 1 + 2(\sigma+\tau) + \sigma\tau = 1 + 2\mu(\nu_1+\nu_2) + \mu^2\nu_1\nu_2,$$

and the constants E and H are as

$$E = -(\nu_1 + \nu_2 + 2\mu\nu_1\nu_2)$$

$$H = -\big(2\mu + \mu^2(\nu_1+\nu_2)\big)$$

We illustrated the results of the theorem 5.1 in the cases of $k = 3,4$. We referred to the homogeneous problem and by invoking the homogeneous conditions we knew that the seeking solution must be divisible by $(1 - z\bar{z})^2$. We supposed that the function $u_0(z,\bar{z}) = \alpha(1 - z\bar{z})^2$, $\alpha \neq 0$ is a solution of the homogeneous problem and this function satisfies the homogeneous boundary conditions, then

$$P_3(\sigma,\tau)4\alpha = 0$$

So the function u_0 is non-zero solution of the homogeneous problem if and only if $P_3(\sigma,\tau) = 0$.

Section 4.4. Some Mathematica programing has been introduced to demonstrate the numerical solution.

Section 4.5. In the final part we review and consider the case of improperly elliptic equation. The consideration are similar to the case of the section 4.2 We represent this equation in the form of complex variables as

$$\left(\frac{\partial}{\partial \bar{z}} - \mu_0 \frac{\partial}{\partial z}\right)^2 \left(\frac{\partial}{\partial \bar{z}} - \mu_1 \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial \bar{z}} - \mu_2 \frac{\partial}{\partial z}\right) u = 0, \tag{4.5.1}$$

$$\mu_k \neq \mu_j \,, 0 < |\mu_j| < 1,$$

with Dirichlet boundary conditions

$$\left.\frac{\partial u}{\partial \bar{z}^{1-j}\, \partial z^j}\right|_\Gamma = F_j(x,y)\,, (x,y) \in \Gamma\,, j = 0,1, \tag{4.5.2}$$

$$u(1,0) = f_0(1), u_r(1,0) = f_1(1), u_\theta(1,0) = f_0(1).$$

Theorem 4.4.1. Assume that $|\mu_0| > |\mu_j|, j = 1,2,$ and the boundary functions F_j belong to the class $A^{(1,\alpha)}(|\mu_0|)$, the class of functions, analytic in the annulus $\{|\mu_0|, |z| < 1\}$, and with first order derivatives Holder continuous up to the boundary, then the problem (4.5.1) and (4.5.2) is uniquely solvable if and only if the following conditions hold.

$$P_j(\sigma,\tau) = \sum_{k=1}^{j-1} \sum_{l=0}^{k-1} (k-l)(\sigma\tau)^l \sum_{m=0}^{k-l-1} \tau^m \sigma^{k-l-m-1} \,, j = 3,4, \ldots \tag{4.5.11}$$

where

$$\sigma = \frac{\mu_1}{\mu_0}, \qquad \tau = \frac{\mu_2}{\mu_0}, \qquad |\sigma| \neq |\tau|.$$

If the conditions (4.5.11) fail, then homogenous problem has finite number of linearly independent solutions, and for $F_j \in A^{(1,\alpha)}(|\mu_0|)$ inhomogeneous problem has a solution if and only if the finite number of linearly independent solvability conditions to boundary functions F_j hold. These defect numbers are equal to the quantity of numbers j, for which the conditions (4.5.11) fail.

Proposition. We consider the problem (4.5.1), (4.5.2) when $\mu_2 = e^{i\alpha}\mu_1$ and $|\mu_0| > |\mu_1|$. In this case we get the result similar to theorem (4.4.1), where instead of P_j from (4.5.11) we have

$$Q_j(z) = z^{j-2}\sum_{p=1}^{j-1} p\sin\left(\frac{p\alpha}{2}\right)\frac{z^{j-p} - z^{p-j}}{z - z^{-1}}\ ,\ \ z = \sigma e^{\frac{i\alpha}{2}}$$

Theorem 4.4.2. Assume that $|\mu_0| < |\mu_1| \le |\mu_2|$, $(\mu_1 \ne \mu_2)$, and the boundary functions F_j belong to the class of $A^{(1,\alpha)}(|\mu_1|)$, then the problem (4.5.1) and (4.5.2) is uniquely solvable if and only if the conditions (4.5.11) hold. If the conditions (4.5.11) fail, then homogeneous problem has finite number of linearly independent solutions, and for $F_j \in A^{(1,\alpha)}(|\mu_1|)$ inhomogeneous problem has a solution if and only if the finite number of linearly independent solvability conditions for boundary conditions F_j hold. Hence defect numbers are equal to the quantity of numbers j, which the conditions (4.5.11) fail.

Theorem 4.4.3. If $\mu_1 = e^{i\alpha}\mu_0\ , \mu_2 = e^{i\beta}\mu_0$, when $\alpha = \frac{2\pi}{l}, \beta = \frac{2\pi}{n}$ and $l, n \in \mathbb{N}$, then the homogeneous problem (4.5.1), (4.5.2) has infinitely many linearly independent solutions. These solutions are polynomials of order $kj_0 + 1$ $(j_0 = [l, n])$.

Theorem 4.4.4. We coincide the inhomogeneous problem (4.5.1), (4.5.2) if

$$\mu_1 = e^{i\alpha}\mu_0, \qquad \mu_2 = e^{i\beta}\mu_0,$$

and $\alpha = \frac{2\pi}{l}, \beta = \frac{2\pi}{n}$ where $l, n \in \mathbb{N}$. Then for solvability of the problem (4.5.1), (4.5.2), it is necessary infinitely many conditions.

List of publications

1. *M. H. Mohammadi*, Mathematical Modeling of Heat Transfer and Transport Phenomena in Three-Dimension with Stefan Free Boundary, Advances and Applications in Fluid Mechanics, Vol. 19, No. 1, pp. 23-34, 2016.

2. *M. H. Mohammadi*, Three-Dimensional Mathematical Modeling of Heat Transfer by Stream Function and its Numerical Solution, Far East Journal of Mathematical Sciences(FJMS), Vol. 99, No. 7, pp. 969-981, 2016.

3. *A. H. Babayan*, M. H. Mohammadi, on the Mathematical Modeling of Garnissage Furnace, ՀԱՊՀ, Լրաբեր, Գիտական հոդվածների ժողովածու, Մաս 1, էջ. 7-16, 2016.

4. *A. H. Babayan*, M. H. Mohammadi, on a Dirichlet Problem for One Properly Elliptic Equation in the Unit Disk, Reports of NAS Armenii, v. 117, No. 3, pp. 192-200, 2017.

5. *M. H. Mohammadi*, Finite Element Method for Transport Phenomena in the Three-Dimensional Case. Բանբեր (ՀԱՊՀ). Տեղեկատվական տեխնոլոգիաներ, էլեկտրոնիկա, ռադիոտեխնիկա, հ. 2, pp. 76-86, 2017.

6. *M. H. Mohammadi*, Numerical Analysis of Fluid Flow and Heat Transfer Based on the Cylindrical Coordinate System. Fluid Mechanics, Vol.4, No.1, pp. 1-13, 2018.

CONCLUSION

Mohammad Hassan Mohammadi

Boundary Value Problems for Elliptic and Parabolic Equations and Applications in Mathematical Modeling of Two-Phase Substance

In the thesis the following results are obtained:

1. Mathematical modeling of heat transfer (Transport Phenomena) in three-dimension for the especial furnaces with cylindrical shapes is derived, where these Furnaces in the business area are called Garnissage furnace. In the present work we obtained the mathematical modeling based on the stream functions and cylindrical coordinates. We prepared the details of the modeling by invoking the conservation laws, and the Stefan condition is applied to describe the behavior of the free boundary.

2. We defined stream function in two dimension and then we gained the mathematical modeling of heat transfer according to stream function. The modeling based on stream function has its sufficient advantages and we prepared complete description about the modeling process, variational approach and weak formulation, and numerical solution of the transport system in two-dimension.

3. We achieved the mathematical modeling in the cylindrical coordinate system according to Garnissage tank shape and its symmetric properties. We got the conservation equations in the cylindrical coordinate system, so the Stefan condition, and then we followed the variational approach to

convert the system of equations into the weak formulation. For expressing the system in the variational formulation we will exert test function with compact support (three-dimensional analog of the Courant function).

4. To complete the finite element method, we discrete the domain to special mesh cubes and we divided them into 24 tetrahedrons. We computed the piecewise linear test function according to 24 tetrahedrons, and then we determined values of coefficients in the linear and nonlinear system. We solved the system numerically by the Newton's method and we exhibited the final results by the corresponding graphs.

5. We focused on the Dirichlet problem for the fourth order elliptic equation with constant coefficients in the unit disc. The case of properly elliptic equation is considered. The characteristic equation has one double root in upper half-plane and two different roots in the lower half-plane. The solution must be found in the class of functions Hölder continuous with first order derivatives up to the boundary. We obtained the new formula for the determination of the defect numbers. The solvability conditions of inhomogeneous problem and linearly independent solutions of homogeneous problem are obtained in explicit form. We demonstrated that the defect numbers may be zero, one or two. The numerical calculation of the defect numbers was done by applying Mathematica programing. The case of improperly elliptic equation is investigated also (characteristic equation has one double root and two simple roots in the upper half-plane). In this case the class of boundary functions, necessary for correctness of the problem was obtained. The formulas for defect numbers was found in this case also. We determine the conditions to the coefficients of the equation, provided infinitely many linearly independent solutions of the homogeneous problem.

ЗАКЛЮЧЕНИЕ

Мохаммад Хассан Мохаммади

Граничные задачи для эллиптических и параболических уравнений и применение в математическом моделировании двухфазной среды

В работе получены следующие результаты:

1. Реализовано математическое моделирование процесса передачи тепла (явление переноса) в трехмерном случае в печи для стекловарения цилиндрической формы (гарниссажной печи). Математическое моделирование основано на использовании функций тока, а также цилиндрических координат. Основные уравнения получены из

законов сохранения, а поведение свободной границы описывается условием Стефана.

2. Определена двумерная функция тока и, используя эту функцию построена математическая модель передачи тепла. Моделирование, основанное на использовании функции тока имеет существенные преимущества. Проведено детальное описание построения математической модели, переход к вариационной задаче, а также слабая формулировка полученной задачи и ее численное решение в двумерном случае.

3. В трехмерном случае, учитывая форму гарниссажной печи, строится математическая модель с использованием цилиндрической координатной системы. Получены уравнения сохранения в цилиндрических координатах. На неподвижной части границы ставятся стандартные условия, а на свободной границе - условие Стефана. Далее, полученная система приводится к вариационной задаче. Для реализации метода конечных элементов для решения этой задачи строится базисная функция с компактным носителем (трехмерный аналог функции Куранта).

4. Для реализации метода конечных элементов разбиваем рассматриваемую область на кубы, каждый из которых, в свою очередь, разбивается на 24 тетраэдра, и, затем используем построенную базисную функцию для приведения задачи к системе уравнений, состоящей из линейной и нелинейной частей, для определения коэффициентов, по которым строится приближенное решение задачи. Полученная система решается методом Ньютона и, затем, решение представлено соответствующими графиками.

5. В единичном круге рассмотрена задача Дирихле для эллиптического уравнения четвертого порядка с постоянными коэффициентами. Сначала рассмотрен случай правильно эллиптического уравнения. Предполагаем, что характеристическое уравнение имеет двукратный корень в верхней полуплоскости и два простых корня в нижней полуплоскости. Решение ищем в классе функций, удовлетворяющих условию Гельдера вплоть до границы вместе с производными первого порядка. Получена новая формула для определения дефектных чисел задачи. Условия разрешимости неоднородной задачи, а также, линейно независимые решения однородной задачи получены в явном виде. Показано, что дефектные числа могут принимать значения ноль, один и два. Численное определение дефектных чисел проведено с использованием пакета программ Mathematica. Далее, рассмотрен случай неправильно эллиптического уравнения (характеристическое уравнение имеет один двукратный корень и два простых корня в верхней полуплоскости) . В этом случае определен класс граничных

функций, в котором рассматриваемая задача корректна, а также формула дефектных чисел. Найдены условия на коэффициенты уравнения, при которых однородная задача имеет бесконечное множество линейно независимых решений.

ԵԶՐԱԿԱՑՈՒԹՅՈՒՆ

Մոհամմադ Հասսան Մոհամմադի

Եզրային խնդիրներ էլիպսական և պարաբոլական հավասարումների համար և կիրառություններ երկֆազ միջավայրի մաթեմատիկական մոդելներում

Աշխատանքում ստացվել են հետևյալ արդյունքները:

1. Եռաչափ դեպքում իրականացվեցլ է գլանաձև ապակու մշակման վարարանում (գարնիսաժ վառարանում) ջերմության փոխանցման մաթեմատիկական մոդելավորումը (տեղափոխության երևույթ): Մաթեմատիկական մոդելավորումը հիմնվում է պոտենցիալ ֆունկցիաների և գլանային կոորդինատների օգտագործման վրա: Հիմնական հավասարումները ստացվել են պահպանման օրենքներից իսկ ազատ եզրը նկարագրվում է Ստեֆանի պայմանով:

2. Ներմուծվել է երկչափ պոտենցիալ ֆունկցիա և այս ֆունկցիայի օգնությամբ կառուցվել է ջերմափոխանակության մաթեմատիկական մոդել: Մոդելավորումը, հիմնված պոտենցիալ ֆունկցիայի օգտագործման վրա ունի զգալի առավելություններ: Մանրամասն նկարագրված է մաթեմատիկական մոդելի կառուցումը և անցումը վարիացիոն խնդրին: Այնուհետև օգտագործելով տրված խնդրի թույլ ձևակերպումը ստանում ենք այդ խնդրի թվային լուծումը երկչափ դեպքում:

3. Եռաչափ դեպքում, հաշվի առնելով գարնիսաժ վարարանի ձևը, մաթեմատիկական մոդելը կառուցվում է գլանային կոորդինատական համակարգի օգնությամբ: Պահպանման օրենքները գրվում են գլանային կոորդինատներում: Եզրի անշարժ մասում տրվում են դասկան պայմանները, իսկ ազատ եզրի վրա Ստեֆանի պայմանը: Այնուհետև, ստացված համակարգը բերվում է վարիացիոն խնդրին: Այս խնդրի լուծման համար վերջավոր տարրերի մեթոդի

իրականացման համար կառուցվել է կոմպակտ կրիչով բազիսային ֆունկցիա (Կուրանտի ֆունկցիայի եռաչափ տարբերակ)։

4. Վերջավոր տարրերի մեթոդի իրականացման համար դիտարկվող տիրույթը տրոհվում է խորանարդերի, որոնցից ամեն մեկը իր հերթին տրոհվում է 24 քառանիստների։ Այնուհետև կառուցված բազիսային ֆունկցիան կիրառվում է խնդիրը գծային և ոչ գծային մասերից բաղկացած հավասարումների համակարգին բերելու նպատակով։ Այդ համակարգը կիրառվում է Նյուտոնի մեթոդով գործակիցների որոշման համար, իսկ լուծումը ներկայացված է համապատասխան գրաֆիկներով։

5. Միավոր շրջանում դիտարկված է հաստատուն գործակիցներով չորրորդ կարգի էլիպսական հավասարման համար Դիրիխլեյի խնդիրը։ Սկզբից դիտարկվում է ճշգրիտ էլիպսական հավասարման դեպքը։ Ենթադրվում է, որ բնութագրիչ հավասարումը ունի կրկնապատիկ արմատ վերին կիսահարթությունում և երկու պարզ արմատներ ստորին կիսահարթությունում։ Լուծումը փնտրում ենք այն ֆունկցիաների դասում, որոնք առաջին կարգի ածանցյալների միասին բավարարում են Հելդերի պայմանին ընթուփ մինձև եզրը։ Գտնվել է խնդրի դեֆեկտային թվերի համար նոր բանաձևը։ Անհամասեռ խնդրի լուծելիության պայմանները և համասեռ խնդրի գծորեն անկախ լուծումները գտնվել են բացահայտ տեսկով։ Ցույց է տրվել, որ զրո, մեկ և երկու դեֆեկտային թվերի հնարվոր արժեքները։ Դեֆեկտային թվերի թվային որոշումը իրականցվել է Mathematica ծրագրային փաթետի օգնությամբ։ Դիտարկվել ք նաև ոչճշգրիտ էլիպսական հավասարման դեփքը (բնութագրիչ հավասարումը ունի մեկ կրկնապատիկ արմատ և երկու պարզ արմատներ վերին կիսահարդությունում)։ Աիս դեփքում գտնվել է եզրային ֆունկցիաների դաս, որտեղ տրված խնդիրը կոռեկտ է և դեֆեկտային թվերի հաշվման բանաձևը։ Որոշվել են նաև հավասարման գործակիցների վրա տրվող պահմանները որի դեփքում համասեռ խնդիրը ունի անվերջ թիվով գծորեն անկախ լուծումներ։