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Inverse problems for some differential operators

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Contents

Preface

The current thesis presents some new results in direct and inverse spectral problems for Sturm-Liouville and Dirac differential operators. It consists of two Chapters, and each Chapter has three Sections.

The Chapter 1 is devoted to Inverse Sturm-Liouville Problems (ISLP) on a finite interval.

In spectral theory of differential operators has been profoundly studied the Sturm-Liouville problem (SLP):

$$\ell y \equiv -y'' + q(x)y = \mu y, \qquad x \in (0, \pi), \tag{0.0.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad \alpha \in (0, \pi], \tag{0.0.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \qquad \beta \in [0, \pi), \tag{0.0.3}$$

which we denote by $L(q, \alpha, \beta)$, where $\mu \in \mathbb{C}$ is the spectral parameter and q is a real-valued function. At the same time, by $L(q, \alpha, \beta)$ we also denote the self-adjoint operator, generated by problem (0.0.1)–(0.0.3) in Hilbert space $L^2[0, \pi]$ (see, e.g. [Nai69, Mar77, LS70]).

The first studies of such problems were performed by D. Bernoulli, J. dAlembert, L. Euler in connection with the solution of the problem describing the vibration of a string.

In a series of articles, dating from 1836-37, Sturm [Stu36a, Stu36b] and Liouville [Lio36] created a whole new subject in mathematical analysis. The theory, later known as Sturm-Liouville theory, deals with linear second-order differential equation

$$(k(x)y')' + (r(x)\mu - q(x))y = 0, \qquad x \in (a, b)$$

with the imposed boundary conditions

$$k(a)y'(a) - hy(a) = 0,$$

$$k(b)y'(b) + Hy(b) = 0.$$

Here k, r and q are given functions (usually k and r are positive), h and H are given constants and μ is a parameter. The boundary-value problem only allows non-trivial solutions (eigenfunctions) for certain values (eigenvalues) of μ. The questions studied by Sturm and Liouville (the direct problem) can roughly be divided into three groups:

- the existence and properties of the eigenvalues,
- behaviour of the eigenfunctions,
- expansion of arbitrary function in an infinite series of eigenfunctions.

Of these, Sturm investigated first two and Liouville the third group. Before 1820 the only question taken up in the theory of differential equations had been: given a differential equation find its solution as an analytic expression. For the general equation Sturm could not find such an expression, and the expression found by Liouville, by successive approximation, was unsuited for the investigation of the properties mentioned above, instead, to obtain the information about properties of the solutions from the equation itself. This shows evidence of a new conception of the theory of differential equations characterized by a new kind of question: given a differential equation, investigate some properties of the solution.

Most notable among the properties to be investigated in the early 19-th century was existence. The existence theorem, formulated and proved by Cauchy in 1835-40, was the first to indicate the broader concept of differential equations.

The Sturm-Liouville theory gave the first theorems on eigenvalue problems and as such it occupies a central place in the prehistory of functional analysis. It was, and has remained till this days, of importance in the technical treatment of many concrete problems in pure and applied mathematics and was as such of more than just conceptual importance.

It is known, that the spectrum of the operator $L(q, \alpha, \beta)$ is discrete and consists of countable real, simple eigenvalues (see, e.g. [Mar77, Yur07**?**]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, emphasizing the dependence of μ_n on q, α and β. Let $\varphi(x, \mu) = \varphi(x, \mu, \alpha, q)$ and $\psi(x, \mu) = \psi(x, \mu, \beta, q)$ are the solutions of the equation (0.0.1), which satisfy the initial conditions

$$\varphi(0,\mu,\alpha,q)=\sin\alpha, \qquad \varphi'(0,\mu,\alpha,q)=-\cos\alpha,$$

$$\psi(\pi,\mu,\beta,q)=\sin\beta, \qquad \psi'(\pi,\mu,\beta,q)=-\cos\beta,$$

respectively. The functions $\varphi(x,\mu_n)$ and $\psi(x,\mu_n)$, $n\geq 0$, are the eigenfunctions, corresponding to the eigenvalue μ_n. The squares of the L^2-norm of these eigenfunctions:

$$a_n=a_n(q,\alpha,\beta):=\int_0^\pi \varphi^2(x,\mu_n)dx, \qquad n=0,1,2,\dots,$$
$$b_n=b_n(q,\alpha,\beta):=\int_0^\pi \psi^2(x,\mu_n)dx, \qquad n=0,1,2,\dots,$$

are called norming constants. Since all the eigenvalues are simple, there exist constants $\kappa_n=\kappa_n(q,\alpha,\beta)\neq 0$, $n=0,1,2,\dots$, such that

$$\varphi(x,\mu_n)=\kappa_n\ \psi(x,\mu_n).$$

The sequences $\{\mu_n\}_{n=0}^\infty$, $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$ and $\{\kappa_n\}_{n=0}^\infty$ are called spectral data (besides these, there are other quantities, which are also called spectral data).

In general, the inverse spectral problem is to reconstruct the operator by some spectral data. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over the world.

In the cases $\sin\alpha\neq 0$ or/and $\sin\beta\neq 0$, often as norming constants is being considered the quantities $\tilde{a}_n=\dfrac{a_n}{\sin^2\alpha}$ and $\tilde{b}_n=\dfrac{b_n}{\sin^2\beta}$.

On of the precise state of the inverse problems is as follows.

What kind the sequences $\{\lambda_n^2\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ should be, to be the spectral data of an operator $L(q,\alpha,\beta)$?

Similar question (but without the condition of fixed α and β and for different class of potential q instead of our $q\in L^1_{\mathbb{R}}[0,\pi]$) was considered first by Gelfand and Levitan in work

[GL51] and after in many papers (we refer only some of them: [GL64, Zhi67, IT83]) and this problem called the inverse Sturm-Liouville problem by "spectral function"[1] (see also, e.g. [Lev84, FY01]).

In Section 1.2 we describe the necessary and sufficient conditions for two sequences $\{\mu_n\}_{n=0}^{\infty} = \{\lambda_n^2\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ (the case $\sin\alpha \neq 0$) to be correspondingly the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with real summable potential q and in advance fixed separated boundary conditions. More precisely, we proof the following:

Theorem (1.2.1)**.** *For a real increasing sequence $\{\lambda_n^2\}_{n=0}^{\infty}$ and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ to be spectral data for boundary-value problem $L(q,\alpha,\beta)$ with a $q \in L^1_{\mathbb{R}}[0,\pi]$ and fixed $\alpha,\beta \in (0,\pi)$ it is necessary and sufficient that the following relations hold:*

1) the sequence $\{\lambda_n\}_{n=0}^{\infty}$ has asymptotic form

$$\lambda_n = n + \frac{\omega}{n} + l_n,$$

where $\omega = const$, $l_n = o\left(\frac{1}{n}\right)$, when $n \to \infty$, and the function $l(\cdot)$, defined by formula $l(x) = \sum_{n=1}^{\infty} l_n \sin nx$, is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$,

2) the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ has asymptotic form

$$\tilde{a}_n = \frac{\pi}{2} + s_n,$$

where $s_n = o\left(\frac{1}{n}\right)$, when $n \to \infty$, and the function $s(\cdot)$, defined by formula $s(x) = \sum_{n=1}^{\infty} s_n \cos nx$, is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$,

3)

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha,$$

[1]Let $\varphi(x,\mu)$ is a solution of the equation (0.0.1) with initial conditions $y(0) = \sin\alpha$, $y'(0) = -\cos\alpha$, on the half axis $[0,\infty)$. Let $f(x)$ be a smooth, compactly supported function. Put $\mathcal{F}(\mu) = \int_0^{\infty} f(x)\varphi(x,\mu)dx$. There exists at least one non-decreasing function $\rho(\mu)$, s.t. $\int_0^{\infty} f^2(x)dx = \int_0^{\infty} \mathcal{F}^2(\mu)d\rho(\mu)$. The function $\rho(\mu)$ is called a spectral function. This theorem is due to Weyl [Wey10]. The spectrum of the operator on half axis is called the set of growth points of the function ρ. In our case, the spectrum of $L(q,\alpha,\beta) is the set of all eigenvalues.$

4)

$$\frac{\tilde{a}_0}{\left(\pi\prod_{k=1}^{\infty}\frac{\mu_k-\mu_0}{k^2}\right)^2}-\frac{1}{\pi}+\sum_{n=1}^{\infty}\left(\frac{\tilde{a}_n n^4}{\left(\pi[\mu_0-\mu_n]\prod_{k=1,k\neq n}^{\infty}\frac{\mu_k-\mu_n}{k^2}\right)^2}-\frac{2}{\pi}\right)=-\cot\beta.$$

The ISLP with α and β from $(0,\pi)$ and $q\in L^2_{\mathbb{R}}[0,\pi]$ was investigated in [IMT84], and for the case $\alpha=\pi,\ \beta\in(0,\pi)$ in [DT84], in original statement of a question, and the solution of these problems the authors reduced to analysis of ISLP with Dirichlet boundary conditions ($y(0)=0,\ y(\pi)=0$, which corresponds to $\alpha=\pi,\ \beta=0$), which was in detail investigated in book [PT87]. After, the authors of paper [KC09] were studied the solution of ISLP $L(q,\pi,\beta)$ (i.e. $\alpha=\pi,\beta\in(0,\pi)$) with $q\in L^2_{\mathbb{R}}[0,\pi]$ in terms of eigenvalues and another type of "norming constants" $\{\nu_n\}_{n=0}^{\infty}$ (see (1.3) in [KC09]).

An intensive development of the spectral theory for various classes of differential and integral operators took place in the 20-th century.

Historically, the first work in the theory of inverse spectral problems for Sturm-Liouville operators is due to Ambarzumyan [Amb29]. Consider SLP generated by differential equation (0.0.1) with Neumann boundary conditions

$$y'(0)=0,$$

$$y'(\pi)=0,$$

i.e. $L(q,\pi/2,\pi/2)$. It is easy to calculate, that if $q(x)\equiv 0$, then the eigenvalues of the problem $L(0,\pi/2,\pi/2)$ are $\mu_n=n^2,\ n\geq 0$. He proved the inverse assertion, i.e. if the eigenvalues of the problem $L(q,\pi/2,\pi/2)$ are n^2, then the potential $q\equiv 0$.

Swedish mathematician Borg [Bor46] was the first who paid attention to the importance of Ambarzumyan's result. Borg showed that in general for the boundary-value problem $L(q,\alpha,\beta)$ additional information is required in order to reconstruct the operator uniquely. Therefore, the Ambarzumyan's result was an exception from the rule and for a given spectrum, there exist infinitely many triples (q,α,β), such that the corresponding problems $L(q,\alpha,\beta)$ have the same spectrum. In the same work he showed that two spectra is sufficient for the unique determination of the operator.

The next step in the development of the classical inverse Sturm-Liouville problem was taken by N. Levinson [Lev49]. Levinson, in 1949, proved, that in the class of even[2] Sturm-Liouville operators the spectrum $\{\mu_n\}_{n=0}^{\infty}$ uniquely determines the potential $q(x)$ and the parameter α.

In 1950 V. Marchenko [Mar50], using transformation (transmutation) operators, proved that if two Sturm-Liouville problems on the half-line,

$$-y'' + q_1(x)y = \mu y, \qquad y'(0) - h_1 y(0) = 0,$$

$$-y'' + q_2(x)y = \mu y, \qquad y'(0) - h_2 y(0) = 0$$

have the same spectral function, then $q_1(x) = q_2(x)$ and $h_1 = h_2$.

The results of G. Borg, N. Levinson and V. Marchenko are important and they show the way to further investigation. There are many other statements of uniqueness theorems for inverse Sturm-Liouville problems, see, e.g. [HL78, MR87, GS97, GS00, FY01, Yur07, Har09, WX09, WX12, Yur13, WW16] and the references therein.

Section 1.3 is devoted to the uniqueness theorems for ISLP. We bring various formulations of famous uniqueness theorems, then we give some new statements of uniqueness theorems. A uniqueness theorem with the lowest eigenvalue for Sturm-Liouville problems with arbitrary self-adjoint boundary conditions is proved.

Theorem (1.3.5)**.** *Let q, $q_0 \in L^1_{\mathbb{R}}(0,\pi)$. If*

$$\mu_0(q) - \mu_0(q_0) = \operatorname{ess\,inf} \hat{q} \quad or \quad \mu_0(q) - \mu_0(q_0) = \operatorname{ess\,sup} \hat{q},$$

then $q(x) = q_0(x) + \mu_0(q) - \mu_0(q_0)$ a.e. on $(0,\pi)$.

As a corollary of this Theorem, we find bounds for the lowest eigenvalue $\mu_0(q,\alpha,\beta)$.

Theorem (1.3.6)**.** *Let $\alpha \in (0,\pi]$ and $\beta \in [0,\pi)$ and $q \in L^1_{\mathbb{R}}(0,\pi)$. The lowest eigenvalue $\mu_0(q,\alpha,\beta)$ has the property*

$$\operatorname{ess\,inf} q(x) + \mu_0(0,\alpha,\beta) \leq \mu_0(q,\alpha,\beta) \leq \operatorname{ess\,sup} q(x) + \mu_0(0,\alpha,\beta).$$

[2] Sturm-Liouvile operator $L(q,\alpha,\beta)$ is called even, if $q(x) = q(\pi - x)$ and $\alpha + \beta = \pi$.

Then, we give a new proof of the famous generalization of Ambarzumyan's theorem with one spectrum (see [IMT84, KC09]).

Theorem (1.3.7)**.** *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$*.*

If

$$\mu_n(q,\alpha,\pi-\alpha) = \mu_n(0,\alpha,\pi-\alpha),$$

for all $n \geq 0$*, then* $q(x) \equiv 0$*.*

Then a uniqueness theorem similar to Marchenko's theorem conditioned by inequality.

Theorem (1.3.8)**.** *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$*. If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$\tilde{a}_n(q,\alpha_0,\beta) \geq \tilde{a}_n(q_0,\alpha_0,\beta_0),$$

for all $n \geq 0$*, then* $\beta = \beta_0$ *and* $q(x) \equiv q_0(x)$*.*

Furthermore, other uniqueness theorems conditioned by inequalities are proved (for $\kappa_n, \tilde{b}_n$ and $\varphi(\pi,\mu_n), \psi(0,\mu_n)$), e.g.

Theorem (1.3.10)**.** *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$*. If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$|\kappa_n(q,\alpha_0,\beta)| \geq |\kappa_n(q_0,\alpha_0,\beta_0)|,$$

for all $n \geq 0$*, then* $\beta = \beta_0$ *and* $q(x) \equiv q_0(x)$*.*

The Chapter 2 is devoted to the direct and inverse spectral theory for Canonical Dirac System. The Dirac equation is a relativistic wave equation derived by British physicist Paul Dirac in 1928. It describes all spin-1/2 massive particles such as electrons and quarks for which parity is a symmetry (see [Tha92]). It is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for special relativity in the context of quantum mechanics. It is the system of partial differential equations, where unknown is 4-component vector-function. In the case of spherical-symmetric potential it reduces to ordinary differential system (see below (0.0.4)).

Let E is two dimensional identical matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are well-known Pauli matrices, which have properties $\sigma_k^2 = E$, $\sigma_k^* = \sigma_k$ (self-adjointness) and $\sigma_k \sigma_j = -\sigma_j \sigma_k$ (anti-commutativity), when $k \neq j$, for $k, j = 1, 2, 3$.

Let p and q are real-valued functions. By $L(p, q, \alpha, \beta) = L(\Omega, \alpha, \beta)$ we denote the boundary-value problem for canonical Dirac system (see [Tit61, GL66, LS70, GD75, Mar77]):

$$\ell y \equiv \Big\{ B \frac{d}{dx} + \Omega(x) \Big\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{0.0.4}$$

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad \alpha \in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big], \tag{0.0.5}$$

$$y_1(\pi) \cos \beta + y_2(\pi) \sin \beta = 0, \quad \beta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big], \tag{0.0.6}$$

where

$$B = \frac{1}{i} \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \sigma_2 p(x) + \sigma_3 q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

Matrix-function $\Omega(\cdot)$ is usually called potential function, and λ is a complex (spectral) parameter, $\lambda \in \mathbb{C}$. By the same $L(p, q, \alpha, \beta)$ we also denote a self-adjoint operator, generated by differential expression ℓ in Hilbert space of two component vector-functions $L^2([0, \pi]; \mathbb{C}^2)$ on the domain

$$D = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in AC[0, \pi], (\ell y)_k \in L^2[0, \pi], k = 1, 2;$$

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \ y_1(\pi) \cos \beta + y_2(\pi) \sin \beta = 0 \Big\}$$

where $AC[0, \pi]$ is the set of absolutely continuous functions on $[0, \pi]$ (see, e.g. [Nai69, LS88]). Marchenko (see [Mar72, p.30]) suggested more general definition. He called $By' + \Omega(x) y = \lambda y$

Dirac system, if $B^2 = -E$ and $B\Omega(x) + \Omega(x)B = 0$, i.e. in this definition there is no restriction on the dimensions of matrices B and $\Omega(x)$, moreover, they can be any operators in corresponding Hilbert space. If we take $\frac{1}{i} \cdot \sigma_1$ as matrix B (and this is appeared in works of Tichmarsh, Krein, Levitan), then any 2×2 dimensional matrix-function, which is anti-commutative with B, has a form $\sigma_2 \cdot p(x) + \sigma_3 \cdot q(x)$. Thus, we call canonical Dirac system the system (0.0.4).

Two Dirac (or Sturm-Liouville) operators are said to be isospectral, if they have the same spectrum. The problem of description of all isospectral Sturm-Liouville operators was suggested and solved (for $q \in L^2_{\mathbb{R}}[0, \pi]$) by E. Trubowitz and coauthors in the series of works [IT83, IMT84, DT84, PT87]. The problem of the form (0.0.1)–(0.0.3) was reduced (using the Darboux transformation) to the problem of the form (0.0.1) with the Dirichlet boundary conditions ($y(0) = 0$, $y(\pi) = 0$) (see [IT83, IMT84]). The inverse Dirichlet problem was completely solved in [PT87]. The case when $\alpha = \pi$, $\beta \in (0, \pi)$ was studied in [DT84, KC09]. In [KC09] the authors gave a more explicit characterization of spectral data in the style of [PT87, IT83] and described isospectral problems for fixed β in another way. In work [JL97] Jodeit and Levitan proposed another approach to the description of all isospectral problems. They used the method based on the Gelfand-Levitan integral equation and transformation operators.

For Dirac operators the description of all isospectral operators is given by Harutyunyan in [Har94]. That description has a "recurrent" form, i.e. at first only one norming constant is being changed, while the others stay unchanged, and obtained a new operator which has the same spectrum and the same norming constants except one. Then changing successively each norming constants, all isospectral operators were inferred, which have the given spectum. Note, that each operator is being obtained from the previous operator. This approach Harutyunyan calles "recurrent" description.

In Section 2.1 we give the description of all self-adjoint regular Dirac operators, on $[0, \pi]$, with the same spectrum, in explicit form, i.e. only in terms of normalized eigenfunctions of the initial operator $L(\Omega,\alpha,0)$ and a given sequence from l^2. With this aim we set $T = \{t_k\}_{k\in\mathbb{Z}} \in l^2$ and by $S(x,T)$ denote square matrix

$$S(x) = \left(\delta_{ij} + (e^{t_j}-1)\int_0^x h_i^*(s)h_j(s)ds\right)_{i,j\in\mathbb{Z}}$$

where δ_{ij} is a Kronecker symbol, $h_n(x) = (h_{n_1}(x), h_{n_2}(x))^*$ are normalized eigenfunctions and $*$ is the sign of transponation. By $S_p^{(k)}(x,T)$ we denote a matrix, which is obtained from the matrix $S(x,T)$, when we replace $k-th$ column of $S(x,T)$ by $\{-(e^{t_k}-1)h_{k_p}(x)\}_{k\in\mathbb{Z}}$ column, $p=1,2$. Now we can formulate our result as follow.

Theorem (2.1.4)**.** *Let $T = \{t_k\}_{k\in\mathbb{Z}} \in l^2$ and $p,q \in L^2_{\mathbb{R}}[0,\pi]$. Then the isospectral operator $L(p(T),q(T),\alpha,0)$, corresponding to T, is generated by potential, which is defined by formula*

$$\Omega(x,T) = \Omega(x) + G(x,x,T)B - BG(x,x,T) = \begin{pmatrix} p(x,T) & q(x,T) \\ q(x,T) & -p(x,T) \end{pmatrix},$$

where

$$G(x,x,T) = \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \begin{pmatrix} \det S_1^{(k)}(x,T) \\ \det S_2^{(k)}(x,T) \end{pmatrix} h_k^*(x).$$

In addition, for $p(x,T)$ and $q(x,T)$ we get an explicit representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} \det S_p^{(k)}(x,T) h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} (-1)^{p-1} \det S_p^{(k)}(x,T) h_{k_p}(x).$$

In Section 2.2 we consider singular Dirac operators on whole and half axes. By $L(p,q)$ we denote a self-adjoint operator (see [Nai69]) on whole axis, generated by differential expression ℓ (see (0.0.4)) in Hilbert space of two-component vector-functions $L^2((-\infty,\infty);\mathbb{C}^2)$ on the domain

$$D = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(-\infty,\infty) \cap AC(-\infty,\infty);$$
$$(\ell y)_k \in L^2(-\infty,\infty), k=1,2 \Big\},$$

where $AC(-\infty,\infty) = AC(\mathbb{R})$ is the set of functions, which are absolutely continuous on each finite segment $[a,b] \subset (-\infty,\infty), -\infty < a < b < \infty$. We assume that this operator has purely discrete spectrum, which consists of simple, real eigenvalues, which we denote by $\lambda_n(p,q)$ and enumerate in increasing order.

For $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by $L(p,q,\alpha)$ we denote a self-adjoint operator on half axis, generated by differential expression ℓ (see (0.0.4)) in Hilbert space of two component vector-functions $L^2((0,\infty);\mathbb{C}^2)$ on the domain

$$D_\alpha = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(0,\infty) \cap AC(0,\infty); \right.$$

$$\left. (\ell y)_k \in L^2(0,\infty), k = 1,2;\ y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0 \right\},$$

where $AC(0,\infty)$ is the set of functions, which are absolutely continuous on each finite segment $[a,b] \subset (0,\infty), 0 < a < b < \infty$. Here we also assume that this operator has purely discrete spectrum, which consists of simple, real eigenvalues, which we denote by $\lambda_n(p,q,\alpha)$ and enumerate in increasing order.

We prove that canonical Dirac expression with linear potential

$$\Omega_0(x) = \sigma_3 \cdot x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$$

generates operators on whole and half axises, for which we can find the eigenvalues and eigenfunctions in explicit form. Precisely, if we take the system of Chebyshev-Hermite orthonormal functions

$$\varphi_n(x) = \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}, \qquad n = 0,1,2,\ldots,$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

Then for Dirac operator $L(0,x)$ the vector-functions

$$U_{-n}(x) = \begin{pmatrix} -\varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix}, \qquad U_0(x) = \begin{pmatrix} 0 \\ \varphi_0(x) \end{pmatrix}, \qquad U_n(x) = \begin{pmatrix} \varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix},$$

for $n = 1, 2, \ldots$, are eigenfunctions corresponding to eigenvalues $\lambda_{-n} = -\sqrt{2n}, \lambda_0 = 0, \lambda_n = \sqrt{2n}$. Then we construct perturbations of these operators with in advance partially given spectrum. Then answer the questions, what will happen with the potential $\Omega_0(x)$ if we change spectral data, i.e., if we add or subtract eigenvalues and change the values of norming constants. For instance, if we extract one eigenvalue, e.g. $\lambda_0(0, x, 0)$ we will get the following theorem.

Theorem (2.2.1)**.** *Let $\rho(\lambda)$ is a spectral function of the operator $L(0, x, 0)$. Then the function $\tilde{\rho}(\lambda)$, defined by relation*

$$\tilde{\rho}(\lambda) = \begin{cases} \rho(\lambda), & \lambda \leq \lambda_0, \\ \rho(\lambda) - a_0^{-1}, & \lambda > \lambda_0, \end{cases}$$

where $a_0 = \sqrt{\pi}/2$, i.e.

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) - \frac{1}{a_0}\delta(\lambda - \lambda_0)d\lambda$$

is also spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ generated by the differential expression $\tilde{l} = B\frac{d}{dx} + \tilde{\Omega}(x)$ and the boundary condition (2.2.14)*, for which $\tilde{\rho}(\lambda)$ is spectral function. Wherein, the potential function $\tilde{\Omega}(x)$ is represented by the following formula*

$$\tilde{\Omega}(x) = \begin{pmatrix} 0 & x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2}ds} \\ x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2}ds} & 0 \end{pmatrix}$$

and for the eigenfunctions the following formulae hold

$$\tilde{V}_n(x) = \begin{pmatrix} V_{n,1}(x) \\ V_{n,2}(x) + \dfrac{e^{-\frac{x^2}{2}} \int_0^x e^{-\frac{s^2}{2}} V_{n,2}(s)ds}{a_0 - \int_0^x e^{-s^2}ds} \end{pmatrix}, \qquad n \in \mathbb{Z}\backslash\{0\}.$$

Definition Let g is defined on (a, b), where $-\infty \leq a < b \leq \infty$. The derivative of function f with respect to function g is called a function $\frac{\partial f}{\partial g(x)}$, which satisfies the equation

$$\frac{d}{d\epsilon} f(g + \epsilon v)\Big|_{\epsilon=0} = \int_a^b \frac{\partial f}{\partial g(x)} v(x)dx,$$

for all $v \in L^2_{\mathbb{R}}(a,b)$.

Section 2.3 is devoted to Dirac operators, which have discrete spectrum. The concept of eigenvalues' gradient is given (for both $L(p,q,\alpha,\beta)$ and $L(p,q,\alpha)$)

$$grad\lambda_n = \left(\frac{\partial\lambda_n}{\partial\alpha}, \frac{\partial\lambda_n}{\partial\beta}, \frac{\partial\lambda_n}{\partial p(x)}, \frac{\partial\lambda_n}{\partial q(x)}\right).$$

and formulae for this gradients are obtained in terms of normalized eigenfunctions.

Theorem (2.3.1)**.** *Let λ_n and $h_n(x)$ are eigenvalues and normalized eigenfunctions of the problem $L(p,q,\alpha,\beta)$, correspondingly. Then the following relations are valid:*

$$\begin{aligned}
\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial\alpha} &= -|h_n(0)|^2,\\
\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial\beta} &= |h_n(\pi)|^2,\\
\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial p(x)} &= |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2,\\
\frac{\partial\lambda_n(\alpha,\beta,p,q)}{\partial q(x)} &= 2h_{n_1}(x)\cdot h_{n_2}(x).
\end{aligned}$$

Similar formulae for operator $L(p,q,\alpha)$.

The concept of eigenvalues' derivative with respect to a canonical matrix-potential is introduced:

$$\frac{\partial\lambda_n}{\partial\Omega(x)} := \begin{pmatrix} \frac{\partial\lambda_n}{\partial p(x)} & \frac{\partial\lambda_n}{\partial q(x)} \\ \frac{\partial\lambda_n}{\partial q(x)} & -\frac{\partial\lambda_n}{\partial p(x)} \end{pmatrix}$$

and shown how it is used to describe the isospectral operators or when finite number of spectral data is changed.

CHAPTER 1

Sturm-Liouville operator on a finite interval

1.1 Introduction and preliminary results

Let $L(q, \alpha, \beta)$ denotes the Sturm-Liouville boundary-value problem (SLP)

$$\ell y \equiv -y'' + q(x)y = \mu y, \qquad x \in (0, \pi), \tag{1.1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad \alpha \in (0, \pi], \tag{1.1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \qquad \beta \in [0, \pi), \tag{1.1.3}$$

where $\mu \in \mathbb{C}$ is a spectral parameter and q is a real-valued, summable function, $q \in L^1_{\mathbb{R}}(0, \pi)$. At the same time, $L(q, \alpha, \beta)$ denotes the self-adjoint operator, generated by problem (1.1.1)–(1.1.3) in Hilbert space $L^2[0, \pi]$ (see, e.g. [Nai69, Mar77, LS88]). We are interested in finding non-trivial solutions of the boundary value problem (1.1.1)-(1.1.3).

Definition 1.1.1. *The values of the parameter μ for which the problem $L(q, \alpha, \beta)$ has non-trivial solutions are called eigenvalues, and the corresponding non-trivial solutions are called eigenfunctions.*

It is known, that under the above conditions the spectrum of operator $L(q, \alpha, \beta)$ is discrete and consists of real, simple eigenvalues (see, e.g. [Mar77, Yur07, Har14]), which we denote by $\mu_n = \mu_n(q, \alpha, \beta)$, $n \geq 0$, emphasizing the dependence of μ_n on q, α and β. We assume that eigenvalues are enumerated in the increasing order, i.e.,

$$\mu_0(q, \alpha, \beta) < \mu_1(q, \alpha, \beta) < \cdots < \mu_n(q, \alpha, \beta) < \cdots .$$

Let $\varphi(x,\mu) = \varphi(x,\mu,\alpha,q)$ and $\psi(x,\mu) = \psi(x,\mu,\beta,q)$ are the solutions of the equation (1.1.1), which satisfy the initial conditions

$$\varphi(0,\mu,\alpha,q) = \sin\alpha, \qquad \varphi'(0,\mu,\alpha,q) = -\cos\alpha, \tag{1.1.4}$$

$$\psi(\pi,\mu,\beta,q) = \sin\beta, \qquad \psi'(\pi,\mu,\beta,q) = -\cos\beta, \tag{1.1.5}$$

respectively. The eigenvalues $\mu_n = \mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots,$ of $L(q,\alpha,\beta)$ are the zeros of characteristic functions

$$\Phi(\mu) = \Phi(\mu,\alpha,\beta) := \varphi(\pi,\mu,\alpha)\cos\beta + \varphi'(\pi,\mu,\alpha)\sin\beta,$$

or

$$\Psi(\mu) = \Psi(\mu,\alpha,\beta) := \psi(0,\mu,\beta)\cos\alpha + \psi'(0,\mu,\beta)\sin\alpha.$$

In virtue of well-known Liouville formula (see, e.g., [CL55, Pon65]), the wronskian $W(x) = W[\varphi,\psi] = \varphi(x)\psi'(x) - \varphi'(x)\psi(x)$ of the solutions $\varphi(x,\mu)$ and $\psi(x,\mu)$ is constant with respect to x. It follows that $W(0) = W(\pi)$ and, consequently $\Phi(\mu,\alpha,\beta) = -\Psi(\mu,\alpha,\beta)$.

It is easy to see that functions $\varphi(x,\mu_n) := \varphi(x,\mu_n,\alpha,q)$ and $\psi(x,\mu_n) := \psi(x,\mu_n,\beta,q)$, $n = 0,1,2,\ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. The squares of the L^2-norm of these eigenfunctions:

$$a_n = a_n(q,\alpha,\beta) := \int_0^\pi \varphi^2(x,\mu_n)dx, \qquad n = 0,1,2,\ldots,$$
$$b_n = b_n(q,\alpha,\beta) := \int_0^\pi \psi^2(x,\mu_n)dx, \qquad n = 0,1,2,\ldots,$$

are called norming constants.

When $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, i.e. $\alpha,\beta \in (0,\pi)$, then the boundary conditions (1.1.2)–(1.1.3) can be written in the following form

$$y(0)\cot\alpha + y'(0) = 0, \tag{1.1.6}$$

$$y(\pi)\cot\beta + y'(\pi) = 0. \tag{1.1.7}$$

In this case $u(x,\mu) = u(x,\mu,\alpha,q)$ and $v(x,\mu) = v(x,\mu,\beta,q)$ denote the solutions of the equation (1.1.1), which satisfy the initial conditions

$$u(0,\mu,\alpha,q)=1, \qquad u'(0,\mu,\alpha,q)=-\cot\alpha, \tag{1.1.8}$$

$$v(\pi,\mu,\beta,q)=1, \qquad v'(\pi,\mu,\beta,q)=-\cot\beta, \tag{1.1.9}$$

respectively. The eigenvalues $\mu_n=\mu_n(q,\alpha,\beta)$, $n=0,1,2,\dots$, of $L(q,\alpha,\beta)$ are the zeros of characteristic function

$$\Delta(\mu):=u(\pi,\mu,\alpha)\cot\beta+u'(\pi,\mu,\alpha)=-[v(0,\mu,\beta)\cot\alpha+v'(0,\mu,\beta)].$$

It is proved that the spectrum uniquely determines the characteristic function $\Delta(\mu)$ by the formula (see, e.g., [FY01, Yur07, Har10a, IT83, JL97])

$$\Delta(\mu)=\pi(\mu_0-\mu)\prod_{n=1}^{\infty}\frac{\mu_n-\mu}{n^2}. \tag{1.1.10}$$

Norming constants, i.e. the squares of the L^2-norm of eigenfunctions $u(x,\mu_n)$ and $v(x,\mu_n)$, are

$$\tilde{a}_n=\tilde{a}_n(q,\alpha,\beta):=\int_0^{\pi}u^2(x,\mu_n)dx, \qquad n=0,1,2,\dots, \tag{1.1.11}$$

$$\tilde{b}_n=\tilde{b}_n(q,\alpha,\beta):=\int_0^{\pi}v^2(x,\mu_n)dx, \qquad n=0,1,2,\dots, \tag{1.1.12}$$

respectively. Since all the eigenvalues are simple, there exist constants $\kappa_n=\kappa_n(q,\alpha,\beta)\neq 0$, $n=0,1,2,\dots$, such that

$$u(x,\mu_n)=\kappa_n\, v(x,\mu_n). \tag{1.1.13}$$

The sequences $\{\mu_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\kappa_n\}_{n=0}^{\infty}$ are called spectral data (besides these, there are other quantities, which are also called spectral data). The inverse Sturm-Liouville problem (ISLP) is to reconstruct the quantities $q,\ \alpha,\ \beta$ by some spectral data.

Lemma 1.1.1. *When $\alpha,\beta\in(0,\pi)$, the following relations hold*

$$\tilde{a}_n=\kappa_n^2\,\tilde{b}_n, \tag{1.1.14}$$

$$u(\pi,\mu_n)=v^{-1}(0,\mu_n)=\kappa_n, \tag{1.1.15}$$

$$\tilde{a}_n=-\kappa_n\,\dot{\Delta}(\mu_n), \tag{1.1.16}$$

for all $n\geq 0$, where $\dot{\Delta}(\mu_n)=\dfrac{d}{d\mu}\Delta(\mu)$.

Proof. The relations (1.1.14) and (1.1.15) easily comes out from relation (1.1.13), initial conditions (1.1.8)–(1.1.9) and the definition of norming constants (1.1.11)–(1.1.12).

It remains to prove the relation (1.1.16). Since

$$-u''(x,\mu)+q(x)u(x,\mu)=\mu u(x,\mu),$$

$$-v''(x,\mu_n)+q(x)v(x,\mu_n)=\mu_n v(x,\mu_n),$$

then, multiplying the first equation by $v(x,\mu_n)$ and the second equation by $u(x,\mu)$, and subtracting the second one from the first, we obtain

$$-\frac{d}{dx}\left[u'(x,\mu)v(x,\mu_n)-u(x,\mu)v'(x,\mu_n)\right]=(\mu-\mu_n)u(x,\mu)v(x,\mu_n).$$

Integrate from 0 to π, we get

$$-u'(\pi,\mu)v(\pi,\mu_n)+u(\pi,\mu)v'(\pi,\mu_n)-u'(0,\mu)v(0,\mu_n)+u(0,\mu)v'(0,\mu_n)=$$
$$=(\mu-\mu_n)\int_0^\pi u(x,\mu)v(x,\mu_n)dx.$$

hence

$$-u'(\pi,\mu)-u(\pi,\mu)\cot\beta+v(0,\mu_n)\cot\alpha+v'(0,\mu_n)=$$
$$=-\Delta(\mu)+\Delta(\mu_n)=(\mu-\mu_n)\int_0^\pi u(x,\mu)v(x,\mu_n)dx.$$

When $\mu\to\mu_n$, we get

$$-\dot{\Delta}(\mu_n)=\int_0^\pi u(x,\mu_n)v(x,\mu_n)dx.$$

Using (1.1.11) and (1.1.13) we arrive at (1.1.16). □

Lemma 1.1.2. *Eigenfunctions related to different eigenvalues are orthogonal in* $L^2(0,\pi)$.

Lemma 1.1.3.

i) The system of eigenfunctions $\{u(x,\lambda_n)\}_{n=0}^\infty$ *of the boundary value problem* $L(q,\alpha,\beta)$ *is complete in* $L^2(0,\pi)$.

ii) Let $f(x)$ *be an absolutely continuous function on* $[0,\pi]$. *Then*

$$f(x)=\sum_{n=0}^\infty \alpha_n u(x,\lambda_n),\quad \alpha_n=\frac{1}{\tilde{a}_n}\int_0^\pi f(t)u(t,\lambda_n)dt, \tag{1.1.17}$$

and the series converges uniformly on $[0,\pi]$.

iii) For $f \in L^2(0,\pi)$, the series (1.1.17) converges in $L^2(0,\pi)$, and the Parseval's equality holds

$$\int_0^\pi |f(t)|^2 dt = \sum_{n=0}^\infty \tilde{a}_n |\alpha_n|^2.$$

The proof of Lemmas 1.1.2–1.1.3 can be found, e.g., in [LS88, FY01].

Definition 1.1.2. *Two operators $L(q,\alpha,\beta)$ and $L(q_0,\alpha_0,\beta_0)$ are called isospectral, if they have the same spectrum, i.e. $\mu_n(q,\alpha,\beta) = \mu_n(q_0,\alpha_0,\beta_0)$, $n \geq 0$.*

Let $L = L(q,\alpha,\beta)$ and $L_0 = L(q_0,\alpha_0,\beta_0)$ be two operators. In what follows, if a certain symbol γ denotes an object related to L, then γ_0 (or γ^0, depending on situation) will denote a similar object related to L_0, and $\hat{\gamma} = \gamma - \gamma_0$.

The problem of description of all problems of the form (1.1.1)–(1.1.3) that have the same spectrum was suggested and solved (for $q \in L^2_{\mathbb{R}}[0,\pi]$) by E. Trubowitz and coauthors in the series of works [IT83, IMT84, DT84, PT87]. The problem of the form (1.1.1), (1.1.6)–(1.1.7) was reduced (using the Darboux transformation) to the problem of the form (1.1.1) with the Dirichlet boundary conditions ($y(0) = 0$, $y(\pi) = 0$) (see [IT83, IMT84]). The inverse Dirichlet problem was completely solved in [PT87]. The case when $\alpha = \pi$, $\beta \in (0,\pi)$ was studied in [DT84, KC09]. In [KC09] the authors gave a more explicit characterization of spectral data in the style of [PT87, IT83] and parameterized isospectral problems for fixed β in another way.

In work [JL97] Jodeit and Levitan proposed another approach to the description of all isospectral problems, using the Gelfand-Levitan integral equation and transformation operators. The authors solved the problem when $q' \in L^2_{\mathbb{R}}[0,\pi]$ and $\alpha,\beta \in (0,\pi)$ (with some remarks for the other cases at the end of the paper).

Let and operator $L_0 = L(q_0,\alpha_0,\beta_0)$ is given, with eigenvalues $\mu_n^0 = \mu_n(q_0,\alpha_0,\beta_0)$ and eigenfunctions $u_0(x,\mu_n^0)$. They constructed the kernel $F(x,y)$ of the integral equation as follows. Let c_n, $n \geq 0$, be arbitrary real numbers, converging to zero, as $n \to \infty$, so rapidly, that the function

$$F(x,y) = \sum_{n=0}^{\infty} c_n u_0(x,\mu_n^0) u_0(y,\mu_n^0) \tag{1.1.18}$$

is continuous and all the second order partial derivatives are also continuous. The integral equation

$$K(x,y) + F(x,y) + \int_0^x K(x,t)F(t,y)dt = 0, \qquad 0 \le y \le x \le \pi, \tag{1.1.19}$$

is called Gelfand-Levitan integral equation[1].

They proved, that if $1 + c_n \tilde{a}_n^0 > 0$, for all $n \geq 0$, then the integral equation (1.1.19) has a unique solution $K(x,y)$ and the function

$$u(x,\mu) = u_0(x,\mu) + \int_0^x K(x,t)u_0(t,\mu)dt$$

is a solution of the differential equation (1.1.1), with potential function

$$q(x) = q_0(x) + 2\frac{d}{dx}K(x,x), \tag{1.1.20}$$

and $u(x,\mu)$ satisfies the initial conditions

$$u(0,\mu) = 1, \qquad u'(0,\mu) = -\cot\alpha,$$

where

$$\cot\alpha = \cot\alpha_0 + \sum_{n=0}^{\infty} c_n. \tag{1.1.21}$$

It means, that the function $u(x,\mu)$ satisfies the initial conditions (1.1.8) for all $\mu \in \mathbb{C}$.

Now $\beta \in (0,\pi)$ should be found, such that $\mu_n(q,\alpha,\beta) = \mu_n(q_0,\alpha_0,\beta_0)$, for all $n \geq 0$, i.e. $u(x,\mu)$ should satisfy, at the point $x = \pi$, the initial conditions (1.1.9)

$$u(\pi,\mu_n^0)\cot\beta + u'(\pi,\mu_n^0) = 0.$$

Such β (in [JL97]) is found from the following relation

$$\cot\beta = \cot\beta_0 + \sum_{n=0}^{\infty} \frac{c_n u_0^2(\pi,\mu_n^0)}{1 + c_n \tilde{a}_n^0}. \tag{1.1.22}$$

[1] Here $F(x,y)$ is a kernel of integral equation (1.1.19), where x is a parameter, $F(x,y)$ is known function and $K(x,y)$ is unknown function, as functions of y.

Thus Jodeit and Levitan showed, that each admissible sequence $\{c_n\}_{n=0}^{\infty}$ generate an isospectral operator $L(q, \alpha, \beta)$, where q, α and β are given by the formulae (1.1.20), (1.1.21) and (1.1.22) respectively. In this way they obtained all the potentials q, with $q' \in L^2(0, \pi)$, having a given spectrum $\{\mu_n^0\}_{n=0}^{\infty}$.

1.2 ISLP with fixed boundary conditions

In this section we consider the boundary-value problem (1.1.1), (1.1.6)–(1.1.7) with $q \in L^1_{\mathbb{R}}[0,\pi]$ and with $\alpha, \beta \in (0,\pi)$. It is connected with the circumstance, that in this case the principle term of asymptotics of $\lambda_n = \sqrt{\mu_n}$ (see below (1.2.1a), (1.2.1b)) is n and the principle term of asymptotics of norming constants $\tilde{a}_n$ (see below (1.2.2a), (1.2.2b)) is $\dfrac{\pi}{2}$. The other three cases: 1) $\alpha = \pi, \beta \in (0,\pi)$, 2) $\alpha \in (0,\pi), \beta = 0$, 3) $\alpha = \pi, \beta = 0$, need a separate investigation and we do not concern it here.

The famous theorem of Marchenko (see [Mar50, Mar52]) asserts that two sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ (or $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{b}_n\}_{n=0}^{\infty}$) uniquely determine the problem $L(q,\alpha,\beta)$.

In this section we state the question:

> What kind the sequences $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ should be, to be the spectral data for a problem $L(q,\alpha,\beta)$ with a $q \in L^1_{\mathbb{R}}[0,\pi]$ and in advance fixed $\alpha,\ \beta \in (0,\pi)$?

Such a question (but without the condition of fixed α and β and for a different class of potential q instead of our $q \in L^1_{\mathbb{R}}[0,\pi]$) was considered first by Gelfand and Levitan in work [GL51] and after in many papers (we refer only some of them: [GL64, Zhi67, IT83]). This problem called inverse Sturm-Liouville problem by "spectral function" (see also [Lev84, FY01]).

Our answer to above question is in the following assertion.

Theorem 1.2.1. *For a real increasing sequence $\{\lambda_n^2\}_{n=0}^{\infty}$ and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ to be spectral data for boundary-value problem $L(q,\alpha,\beta)$ with a $q \in L^1_{\mathbb{R}}[0,\pi]$ and fixed $\alpha, \beta \in (0,\pi)$ it is necessary and sufficient that the following relations hold:*

1) the sequence $\{\lambda_n\}_{n=0}^{\infty}$ has asymptotic form

$$\lambda_n = n + \frac{\omega}{n} + l_n, \tag{1.2.1a}$$

where $\omega = const$,

$$l_n = o\left(\frac{1}{n}\right), \quad when \quad n \to \infty, \tag{1.2.1b}$$

and the function $l(\cdot)$, defined by formula

$$l(x) = \sum_{n=1}^{\infty} l_n \sin nx, \tag{1.2.1c}$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, i.e.

$$l \in AC(0, 2\pi); \tag{1.2.1d}$$

2) the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ has asymptotic form

$$\tilde{a}_n = \frac{\pi}{2} + s_n, \tag{1.2.2a}$$

where

$$s_n = o\left(\frac{1}{n}\right), \quad \text{when} \quad n \to \infty, \tag{1.2.2b}$$

and the function $s(\cdot)$, defined by formula

$$s(x) = \sum_{n=1}^{\infty} s_n \cos nx, \tag{1.2.2c}$$

is absolutely continuous on arbitrary segment $[a, b] \subset (0, 2\pi)$, i.e.

$$s \in AC(0, 2\pi); \tag{1.2.2d}$$

3)

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot \alpha, \tag{1.2.3}$$

4)

$$\frac{\tilde{a}_0}{\left(\pi \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\left(\pi[\mu_0 - \mu_n] \prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi}\right) = -\cot \beta. \tag{1.2.4}$$

In what follows, under condition (1.2.1) we understand the conditions (1.2.1a)–(1.2.1d) and under condition (1.2.2) the conditions (1.2.2a)–(1.2.2d).

To prove Theorem 1.2.1 we use the following assertion, which has an independent interest.

Theorem 1.2.2. *Let $q \in L^1_{\mathbb{R}}[0, \pi]$ and $\alpha, \beta \in (0, \pi)$. Then for norming constants $\tilde{a}_n = \tilde{a}_n(q, \alpha, \beta)$ and $\tilde{b}_n = \tilde{b}_n(q, \alpha, \beta)$ the following relations are valid*

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot \alpha, \tag{1.2.5}$$

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot \beta. \tag{1.2.6}$$

We should say, that the asymptotics (1.2.1) and (1.2.2) have their roots in the paper of Zhikov [Zhi67]. Also we must note that conditions (1.2.4) and (1.2.6) are equivalent. It is a corollary of the fact, that norming constants $\tilde{b}_n = \tilde{b}_n(q, \alpha, \beta)$, $n = 0, 1, 2, \ldots$, can be represented by spectrum $\{\mu_n\}_{n=0}^{\infty}$ and norming constants $\{\tilde{a}_n\}_{n=0}^{\infty}$ by the next way:

From (1.1.10) we know that the specification of the spectrum $\{\mu_n(q, \alpha, \beta)\}_{n=0}^{\infty}$ uniquely determines the characteristic function $\Delta(\mu)$ and, consequently, its derivative $\dot{\Delta}(\mu)$. In particular, if $\alpha, \beta \in (0, \pi)$ the following formulae hold (see, e.g. [Har10a]):

$$\dot{\Delta}(\mu_0) = -\pi \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}, \tag{1.2.7}$$

$$\dot{\Delta}(\mu_n) = -\frac{\pi}{n^2} [\mu_0 - \mu_n] \prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}, \tag{1.2.8}$$

for all $n = 1, 2, \ldots$. On the other hand, from relation (1.1.16) of Lemma 1.1.1, we have

$$\tilde{a}_n = -\kappa_n \dot{\Delta}(\mu_n). \tag{1.2.9}$$

Taking into account the relations (1.1.14) and (1.2.7)-(1.2.9) we find formulae for $1/\tilde{b}_0$ and $1/\tilde{b}_n$ with $n = 1, 2, \ldots$ (in terms of $\{\mu_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$):

$$\frac{1}{\tilde{b}_0} = \frac{\tilde{a}_0}{\pi^2 \left(\prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2} \right)^2}, \tag{1.2.10}$$

$$\frac{1}{\tilde{b}_n} = \frac{\tilde{a}_n n^4}{\pi^2 [\mu_0 - \mu_n]^2 \left(\prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2} \right)^2}, \quad n = 1, 2, \ldots. \tag{1.2.11}$$

So, we can change the second assertion in Theorem 1.2.2 by the assertion

$$\frac{\tilde{a}_0}{\left(\pi \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\left(\pi [\mu_0 - \mu_n] \prod_{k=1, k\neq n}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi} \right) = -\cot \beta.$$

which coincides with (1.2.4).

The ISLP with α and β from $(0, \pi)$ and $q \in L^2_{\mathbb{R}}[0, \pi]$ was investigated in [IMT84], and for the case $\alpha = \pi$, $\beta \in (0, \pi)$ in [DT84], in original statement of a question, and the solution of these problems were reduced to analysis of ISLP with Dirichlet boundary conditions ($y(0) = 0$, $y(\pi) = 0$, which corresponds to $\alpha = \pi$, $\beta = 0$), which was in detail investigated in book [PT87]. We can say, that the problems, solved in [IMT84, DT84], ware not the inverse problems by "spectral function", but these problems are deeply connected with it, and [IMT84, DT84] play an important role in development of inverse problems. After, the authors of paper [KC09] were studied the solution of ISLP for $L(q, \pi, \beta)$ (i.e. $\alpha = \pi, \beta \in (0, \pi)$) with $q \in L^2_{\mathbb{R}}[0, \pi]$ in terms of eigenvalues and "norming constants"[2] $\{\nu_n\}_{n=0}^{\infty}$ (see (1.3) in [KC09]), which they introduced for this case[3]. They proved that the condition (1.2.6) is necessary for norming constants $\tilde{b}_n(q, \pi, \beta)$. They also proved that the conditions (1.2.5) and (1.2.6) are necesssary for norming constants of problem $L(q, \alpha, \beta)$, if $\sin \alpha \neq 0$ and $\sin \beta \neq 0$. Really they formulated these relations in terms of "norming constants" $\{\nu_n\}_{n=0}^{\infty}$, but it is easy to verify that these formulations are equivalent. Theorem 1.2.2 was proved, for the case $q \in L^2_{\mathbb{R}}[0, \pi]$, with different methods, in [KC09] and [AH15]. It is also must be noted, that the relations (1.2.5) and (1.2.6) come from the paper of Jodeit and Levitan [JL97].

Note that the asymptotic behavior of $\{\mu_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ are standard conditions for the solvability of the inverse problem (see, e.g., [GL51, GL64, Zhi67, FY01]). The conditions (1.2.3) and (1.2.4), which we add to the conditions (1.2.1) and (1.2.2), guarantee that α and β, which we construct during the solution of the inverse problem, are the same that we fixed in advance. At the same time Theorem 1.2.2 says that the conditions (1.2.3) and (1.2.4), which equivalent to (1.2.5) and (1.2.6), are necessary.

1.2.1 The kernel of Gelfand-Levitan integral equation

Consider the function $a(x)$, defined as

$$a(x) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x}{\tilde{a}_n} - \frac{\cos nx}{\tilde{a}_n^0} \right), \tag{1.2.12}$$

where $\tilde{a}_0^0 = \pi$, $\tilde{a}_n^0 = \dfrac{\pi}{2}$, for $n = 1, 2, \ldots$.

[2] $\nu_n = \log \left| \dfrac{v(\pi, \mu_n)}{v'(0, \mu_n)} \right|$, $n = 0, 1, 2, \ldots$.

[3] We should say, that this "norming constants" had been introduced before in paper [HN04], and corresponding uniqueness theorem (see theorem 2.3 in [KC09]) had been proved in [HN04] (see theorem 3).

Lemma 1.2.1. *Let the sequences* $\{\lambda_n\}_{n=0}^{\infty}$ *and* $\{\tilde{a}_n\}_{n=0}^{\infty}$ *have the properties* (1.2.1) *and* (1.2.2) *correspondingly. Then* $a \in AC(0, 2\pi)$.

Proof. We set

$$\rho_n = \lambda_n - n = \frac{\omega}{n} + l_n = O\left(\frac{1}{n}\right). \tag{1.2.13}$$

The general term of the sum (1.2.12) can be rewritten as follows

$$\frac{\cos \lambda_n x}{\tilde{a}_n} - \frac{\cos nx}{\tilde{a}_n^0} = \frac{\cos \lambda_n x}{\tilde{a}_n} - \frac{\cos nx}{\tilde{a}_n} + \frac{\cos nx}{\tilde{a}_n} - \frac{\cos nx}{\tilde{a}_n^0} =$$
$$= \frac{1}{\tilde{a}_n}\left(\cos \lambda_n x - \cos nx\right) + \left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right)\cos nx. \tag{1.2.14}$$

So we can rewrite the series (1.2.12) in the following form

$$a(x) = \sum_{n=0}^{\infty} \frac{1}{\tilde{a}_n}\left(\cos \lambda_n x - \cos nx\right) + \sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right)\cos nx. \tag{1.2.15}$$

The difference $(\cos \lambda_n x - \cos nx)$ can be represented as follows

$$\cos \lambda_n x - \cos nx = \cos(n + \rho_n)x - \cos nx =$$
$$= \cos nx \cos \rho_n x - \sin nx \sin \rho_n x - \cos nx =$$
$$= -\cos nx(1 - \cos \rho_n x) - \sin nx \sin \rho_n x =$$
$$= -2\sin^2 \frac{\rho_n x}{2} \cos nx - \rho_n x \sin nx - (\sin \rho_n x - \rho_n x)\sin nx \tag{1.2.16}$$

and for the difference $\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right)$ we have

$$\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0} = \frac{1}{\frac{\pi}{2} + s_n} - \frac{1}{\frac{\pi}{2}} = -\frac{2}{\pi} \cdot \frac{s_n}{1 + \frac{\pi}{2}s_n} = -\frac{2}{\pi} \cdot s_n + q_n, \tag{1.2.17}$$

where $q_n = o\left(\frac{1}{n^2}\right)$. From the latter relation (1.2.17) it follows, that for sufficiently large n we have

$$\frac{1}{\tilde{a}_n} = \frac{1}{\tilde{a}_n^0} + o\left(\frac{1}{n}\right) = \frac{2}{\pi} + o\left(\frac{1}{n}\right). \tag{1.2.18}$$

And then, according to the relations (1.2.13) and (1.2.18), we obtain

$$\frac{1}{\tilde{a}_n}(-\rho_n x)\sin nx = \left[-\frac{2}{\pi} + o\left(\frac{1}{n}\right)\right]\left[\frac{\omega}{n} + l_n\right] x \sin nx =$$
$$= -\frac{2}{\pi}\omega x \frac{\sin nx}{n} - \frac{2}{\pi} l_n x \sin nx + r_n x \sin nx, \tag{1.2.19}$$

where $r_n = o\left(\frac{1}{n^2}\right)$.

Since $\rho_n = O\left(\frac{1}{n}\right)$ and $\sin y - y = O\left(n^3\right)$ for y close to zero, then

$$\sin \rho_n x - \rho_n x = O\left((\rho_n x)^3\right) = O\left(\frac{1}{n^3}\right), \tag{1.2.20}$$

$$\sin^2 \frac{\rho_n x}{2} = O\left(\left(\frac{\rho_n x}{2}\right)^2\right) = O\left(\frac{1}{n^2}\right). \tag{1.2.21}$$

Thus, taking into account the relations (1.2.16)–(1.2.21) we can rewrite the function (1.2.15) as follows

$$a(x) = a_1(x) + a_2(x),$$

where

$$a_1(x) = -\frac{2\omega x}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n} - \frac{2x}{\pi}\sum_{n=1}^{\infty} l_n \sin nx - \frac{2}{\pi}\sum_{n=1}^{\infty} s_n \cos nx,$$

$$a_2(x) = -\sum_{n=1}^{\infty}\frac{1}{\tilde{a}_n}\left(\sin \rho_n x - \rho_n x\right)\sin nx - 2\sum_{n=0}^{\infty}\frac{1}{\tilde{a}_n}\sin^2\frac{\rho_n x}{2}\cos nx +$$
$$+\sum_{n=0}^{\infty} q_n \cos nx + x\sum_{n=1}^{\infty} r_n \sin nx.$$

Since the first sum $\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2}$, for $x \in (0, 2\pi)$, then, in particular, it belongs to $AC(0, 2\pi)$, and the second sum $\sum_{n=1}^{\infty} l_n \sin nx$ also belongs to $AC(0, 2\pi)$ according to the condition (1.2.1d) of Theorem 1.2.1. In its turn the sum $\sum_{n=1}^{\infty} s_n \cos nx$ belongs to $AC(0, 2\pi)$ according to the condition (1.2.2d) of Theorem 1.2.1. The other four sums of $a_2(x)$ converge absolutely and uniformly on $[0, 2\pi]$ and are continuous differentiable functions, and hence belong to $AC(0, 2\pi)$.

These complete the proof. □

Lemma 1.2.2. *Let sequences* $\{\lambda_n\}_{n=0}^{\infty}$ *and* $\{\tilde{a}_n\}_{n=0}^{\infty}$ *have the properties* (1.2.1) *and* (1.2.2), (1.2.3) *correspondingly. Then the function* F, *defined in triangle* $0 \le t \le x \le \pi$ *by formula*

$$F(x,t) = \sum_{n=0}^{\infty}\left(\frac{\cos \lambda_n x \cos \lambda_n t}{\tilde{a}_n} - \frac{\cos nx \cos nt}{\tilde{a}_n^0}\right), \tag{1.2.22}$$

is absolutely continuous function with respect to each variable and function

$$f(x) := \frac{d}{dx}F(x,x),$$

is summable on $(0,\pi)$*, i.e.* $f \in L^1(0,\pi)$*.*

Proof. It is easy to see that

$$F(x,t) = \frac{1}{2}\left[a(x+t) + a(x-t)\right]. \tag{1.2.23}$$

Since, $a \in AC(0,2\pi)$, then we can infer that the function $F(x,t)$ with respect to both of the variables has the same smoothness as $a(x)$. For the function $F(x,x)$ we have

$$F(x,x) = \frac{1}{2}\left[a(2x) + a(0)\right]. \tag{1.2.24}$$

According to (1.2.3) and (1.2.12)

$$a(0) = \sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right) = \cot\alpha,$$

for $\alpha \in (0,\pi)$. Hence $a(0)$ has a sense and, therefore, $F(x,x)$ too. Besides this

$$\frac{d}{dx}F(x,x) = \frac{1}{2}\frac{d}{dx}a(2x), \tag{1.2.25}$$

and since $a \in AC(0,2\pi)$, then the function $a(2x)$ belongs to $AC(0,\pi)$, and its derivative belongs to $L^1_{\mathbb{R}}(0,\pi)$, i.e. the function $\frac{d}{dx}F(x,x)$ belongs to $L^1_{\mathbb{R}}(0,\pi)$. $\square$

1.2.2 Proof of Theorem 1.2.2

The solution u has the well known representation (see [FY01, GL64, GL51, Mar50])

$$u(x,\lambda,\alpha,q) = \cos\lambda x + \int_0^x G(x,t)\cos\lambda t dt,$$

where about the kernel $G(x,t)$ we know (in particular) that

$$G(x,x) = -\cot\alpha + \frac{1}{2}\int_0^x q(s)ds. \tag{1.2.26}$$

It is also known that $G(x,t)$ satisfies to the Gelfand-Levitan integral equation

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \quad 0 \le t \le x, \tag{1.2.27}$$

where (see [FY01])

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{\tilde{a}_n} - \frac{\cos nx \cos nt}{\tilde{a}_n^0} \right) \tag{1.2.28}$$

where $\tilde{a}_0^0 = \pi$ and $\tilde{a}_n^0 = \frac{\pi}{2}$ for $n = 1, 2, \ldots$. From (1.2.26)–(1.2.28) it follows that

$$\begin{aligned} G(0,0) = -F(0,0) &= -\sum_{n=0}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0} \right) \\ = -\left(\frac{1}{\tilde{a}_0} - \frac{1}{\pi} \right) &- \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi} \right) = -\cot \alpha. \end{aligned} \tag{1.2.29}$$

Thus, (1.2.5) is proved.

Let us now consider the functions (compare with [JL97])

$$p(x, \mu_n) = \frac{u(\pi - x, \mu_n, \alpha, q)}{u(\pi, \mu_n, \alpha, q)} = \frac{u(\pi - x, \mu_n)}{u(\pi, \mu_n)}, \quad n = 0, 1, 2, \ldots. \tag{1.2.30}$$

Since $u(x, \mu, \alpha, q)$ satisfies the equation (1.1.1), and

$$p'(x, \mu_n) = -\frac{u'(\pi - x, \mu_n)}{u(\pi, \mu_n)}, \quad p''(x, \mu_n) = \frac{u''(\pi - x, \mu_n)}{u(\pi, \mu_n)},$$

we can see that $p(x, \mu_n)$ satisfy the equation

$$-p''(x, \mu_n) + q(\pi - x)p(x, \mu_n) = \mu_n p(x, \mu_n)$$

and the initial conditions

$$p(0, \mu_n) = 1, \quad p'(0, \mu_n) = -\frac{u'(\pi, \mu_n)}{u(\pi, \mu_n)} = -(-\cot \beta) = -\cot(\pi - \beta). \tag{1.2.31}$$

Also we have

$$\begin{aligned} p(\pi, \mu_n) &= \frac{u(0, \mu_n)}{u(\pi, \mu_n)} = \frac{\sin \alpha}{u(\pi, \mu_n)} = \frac{\sin(\pi - \alpha)}{u(\pi, \mu_n)}, \\ p'(\pi, \mu_n) &= -\frac{u'(0, \mu_n)}{u(\pi, \mu_n)} = -\frac{-\cos \alpha}{u(\pi, \mu_n)} = \frac{-\cos(\pi - \alpha)}{u(\pi, \mu_n)}. \end{aligned}$$

It follows, that $p(x, \mu_n)$ satisfy to the boundary condition

$$p(\pi, \mu_n) \cos(\pi - \alpha) + p'(\pi, \mu_n) \sin(\pi - \alpha) = 0, \quad n = 0, 1, 2, \ldots.$$

Let us denote $q^*(x) := q(\pi - x)$. Since $\mu_n(q^*, \pi-\beta, \pi-\alpha) = \mu_n(q, \alpha, \beta)$ (it is easy to prove and is well known [IT83]), it follows, that $p(x, \mu_n)$, $n = 0, 1, 2, \ldots$, are the eigenfunctions of problem $L(q^*, \pi-\beta, \pi-\alpha)$, which have the initial conditions (1.2.31); i.e. $p(x, \mu_n) = u(x, \mu_n, \pi-\beta, q^*)$, $n = 0, 1, 2, \ldots$.

Thus, as in (1.2.29), norming constants $a_n^* = \|p(\cdot, \mu_n)\|^2$ must satisfy

$$\left(\frac{1}{a_0^*} - \frac{1}{\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{a_n^*} - \frac{2}{\pi}\right) = \cot(\pi - \beta) = -\cot\beta. \tag{1.2.32}$$

For the norming constants a_n^*, using relations (1.1.14), (1.1.15) and (1.2.30), we get

$$\begin{aligned} a_n^* = \int_0^\pi p^2(x, \mu_n)dx = \int_0^\pi \frac{u^2(\pi - x, \mu_n)}{u^2(\pi, \mu_n)}dx \\ = -\frac{1}{u^2(\pi, \mu_n)}\int_\pi^0 u^2(s, \mu_n)ds = \frac{1}{u^2(\pi, \mu_n)}\int_0^\pi u^2(s, \mu_n)ds \\ = \frac{\tilde{a}_n(q, \alpha, \beta)}{u^2(\pi, \mu_n)} = \frac{\tilde{a}_n(q, \alpha, \beta)}{\kappa_n^2(q, \alpha, \beta)} = \tilde{b}_n. \end{aligned}$$

Therefore, we can rewrite (1.2.32) in the following form

$$\left(\frac{1}{\tilde{b}_0} - \frac{1}{\pi}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = \cot(\pi - \beta) = -\cot\beta.$$

This completes the proof.

1.2.3 Proof of the main result

Necessity. If $\{\lambda_n^2\}_{n=0}^\infty$ are the eigenvalues and $\{\tilde{a}_n\}_{n=0}^\infty$ are the norming constants of the problem $L(q, \alpha, \beta)$, then for $\mu_n = \lambda_n^2$ the asymptotics (1.2.1) was proved in [Har16], and for $\tilde{a}_n$ the asymptotics (1.2.2) was proved in [HP16]. The necessity of connections (1.2.3) and (1.2.4) follows from Theorem 1.2.2.

Sufficiency. In [GL64] there is a proof of such assertion:

Theorem 1.2.3 ([GL64]). *For real numbers $\{\lambda_n^2\}_{n=0}^\infty$ and $\{\tilde{a}_n\}_{n=0}^\infty$ to be the spectral data for a certain boundary-value problem $L(q, \alpha, \beta)$ with $q \in L^1_{\mathbb{R}}[0, \pi]$, ($\alpha, \beta \in (0, \pi)$), it is necessary and sufficient that relations* (1.2.1a)–(1.2.1b) *and* (1.2.2a)–(1.2.2b) *hold, and the function $F(\cdot, \cdot)$ has partial derivatives, which are summable with respect to each variable.*

Thus, if we have a real sequence $\{\mu_n\}_{n=0}^{\infty} = \{\lambda_n^2\}_{n=0}^{\infty}$, which has the asymptotic representation (1.2.1a)–(1.2.1b) and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$, which has the asymptotic representation (1.2.2a)–(1.2.2b), then, according to the Theorem 1.2.3, there exist a function $q \in L^1_{\mathbb{R}}[0,\pi]$ and some constants $\tilde{\alpha}, \tilde{\beta} \in (0,\pi)$ such that λ_n^2, $n = 0,1,2,\ldots$, are the eigenvalues and $\tilde{a}_n$, $n = 0,1,2,\ldots$, are norming constants of a Sturm-Liouville problem $L(q,\tilde{\alpha},\tilde{\beta})$.

The function $q(x)$ and constants $\tilde{\alpha}, \tilde{\beta}$ are obtained on the way of solving the inverse problem by Gelfand-Levitan method. The algorithm of that method is as follows:

First we define the function $F(x,t)$ by formula

$$F(x,t) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x \cos \lambda_n t}{\tilde{a}_n} - \frac{\cos nx \cos nt}{\tilde{a}_n^0} \right). \tag{1.2.33}$$

Note that this function is defined by $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\tilde{a}_n\}_{n=0}^{\infty}$ uniquely. Then we solve Gelfand-Levitan integral equation [Mar50, GL51, GL64, FY01]

$$G(x,t) + F(x,t) + \int_0^x G(x,s)F(s,t)ds = 0, \qquad 0 \le t \le x, \tag{1.2.34}$$

where $G(x,\cdot)$ is unknown function. Find function $G(x,t)$, with the help of which we construct a function

$$u(x,\lambda^2) = \cos \lambda x + \int_0^x G(x,t) \cos \lambda t dt, \tag{1.2.35}$$

which is defined for all $\lambda \in \mathbb{C}$. It is proved (see [GL64]) that

$$-u''(x,\lambda^2) + \left(2\frac{d}{dx}G(x,x)\right)u(x,\lambda^2) = \lambda^2 u(x,\lambda^2), \tag{1.2.36}$$

almost everywhere on $(0,\pi)$, and

$$u(0,\lambda^2) = 1,$$

$$u'(0,\lambda^2) = G(0,0).$$

If we denote

$$G(0,0) = -\cot \tilde{\alpha}, \tag{1.2.37}$$

then the solution (1.2.35) of equation (1.2.36) will satisfy the boundary condition

$$u(0,\lambda^2)\cot\tilde{\alpha} + u'(0,\lambda^2) = 0$$

for all $\lambda \in \mathbb{C}$. Since from (1.2.34) follows that $G(0,0) = -F(0,0)$ and from (1.2.33) follows that $F(0,0) = \sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right)$, hence we get

$$\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right) = \cot\tilde{\alpha}. \tag{1.2.38}$$

From the relation (1.2.38) and our condition (1.2.3) on the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ we find that $\tilde{\alpha} = \alpha$.

It is also proved (see, e.g., [GL64]) that the expression

$$\frac{u'(\pi,\lambda_n^2)}{u(\pi,\lambda_n^2)} = \frac{u'(\pi,\lambda_m^2)}{u_n(\pi,\lambda_m^2)}$$

is a constant (i.e. does not depend on n), which we denote by $-\cot\tilde{\beta}$. Thus the functions $u(x,\lambda_n^2), n = 0,1,2,\ldots$, are the eigenfunctions of a problem $L(q,\tilde{\alpha},\tilde{\beta})$, where $q(x) = 2\frac{d}{dx}G(x,x)$, $\tilde{\alpha}$ is in advance given α and we should have $\tilde{\beta}$ equals β. We know from the Theorem 1.2.2, that for problem $L(q,\alpha,\tilde{\beta})$ it holds

$$\frac{1}{\tilde{b}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{b}_n} - \frac{2}{\pi}\right) = -\cot\tilde{\beta}.$$

Thus, if we obtain condition (1.2.6), then we guarantee that $\tilde{\beta} = \beta$. But (1.2.6) deals with the norming constants $\tilde{b}_n$, which are not independent. We have shown, that we can represent $\tilde{b}_n$ by a_n and $\{\mu_k\}_{k=0}^{\infty}$ by formulae (1.2.10), (1.2.11). Therefore, instead of (1.2.6), we obtain the condition in the form (1.2.4).

This completes the proof.

1.3 Uniqueness theorems for ISLP

Historically, the first work in the theory of inverse spectral problems for Sturm-Liouville operators belongs to Ambarzumyan. It is well-known that the eigenvalues of the operator $L(0, \pi/2, \pi/2)$ are n^2, $n \geq 0$. Ambarzumyan proved the inverse assertion.

Theorem 1.3.1 (Ambarzumyan [Amb29])**.** *If*

$$\mu_n(q, \pi/2, \pi/2) = \mu_n(0, \pi/2, \pi/2) = n^2,$$

for all $n \geq 0$, then $q(x) \equiv 0$.

In 1946 Borg [Bor46] showed that in general for one given spectrum, there exist infinitely many triples (q, α, β), such that the corresponding problems $L(q, \alpha, \beta)$ have the same spectrum. In the same work he showed that two spectra uniquely determine the operator.

Let $L = L(q, \alpha, \beta)$ and $L_0 = L(q_0, \alpha_0, \beta_0)$ be two operators. In 1950 Marchenko proved a uniqueness theorem with spectral function for Sturm-Liouville operators on the half-line, using the transformation operator and Parsevals equality. That theorem works also for Sturm-Liouville operators on the finite interval and the theorem of Marchenko, in terms of eigenvalues and norming constants, can be formulated as follows[4].

Theorem 1.3.2 (Marchenko [Mar50, Mar52])**.** *If*

$$\mu_n(q, \alpha, \beta) = \mu_n(q_0, \alpha_0, \beta_0),$$

$$a_n(q, \alpha, \beta) = a_n(q_0, \alpha_0, \beta_0),$$

for all $n \geq 0$, then $q(x) \equiv q_0(x)$ and $\alpha = \alpha_0$, $\beta = \beta_0$.

There are many other statements of uniqueness theorems for inverse Sturm-Liouville problems. Let $\Phi(x, \mu)$ be the solution of equation (1.1.1) under the boundary conditions $\Phi(0) - h\Phi'(0) = 1$ and $\Phi(\pi) + H\Phi'(\pi) = 0$. Set $M(\mu) := \Phi(0, \mu)$. The functions $\Phi(x, \mu)$ and $M(\mu)$ are called the Weyl solution and the Weyl function for the Sturm-Liuoville boundary

[4] The theorem of Marchenko is more general, see e.g. [Mar50, Mar52, Lev62, FY01].

value problem, respectively. The Weyl function was introduced first (for the case of the half-line) by H. Weyl. The specification of the Weyl function uniquely determines the potential function and boundary parameters. In 1978, Hochstadt and Lieberman [HL78] proved that if the potential function q is given on the interval $(0, \pi/2)$, then one spectrum is sufficient to determine q on the interval $(0, \pi)$. After, Gesztesy and Simon (see [GS97, GS00]) gave several generalizations of the Hochstadt-Lieberman theorem where the potential q is known on a larger interval $[0, a]$ with $a \in [\pi/2, \pi]$ and the set of common eigenvalues is sufficiently large. Then, Guangsheng Wei and coauthors in several papers (see, e.g. [WX09, WX12, WW16]) obtained uniqueness results which imply that the potential q can be completely determined even if only partial information is given on q together with partial information on the spectral data, consisting of either one full spectrum and a subset of norming constants or a subset of pairs of eigenvalues and the corresponding norming constants. By McLaughlin and Rundell [MR87] it is shown that a particular eigenvalue for an infinite set of different boundary conditions is sufficient to determine the potential function.

1.3.1 Generalizations of Ambarzumyan's theorem

There are many generalizations of Ambarzumyan's theorem in various directions, we mention several of them (see, e.g. [Kuz62, CA88, CLW01, FY01, YHY10, YW11, Yur13] and references therein).

1.3.1.1 With the lowest eigenvalue

In [FY01], Freiling and Yurko showed that it is not necessary to specify the whole spectrum and it is enough to have information only on the first eigenvalue.

Theorem 1.3.3 (Freiling-Yurko [FY01]). *If*

$$\mu_0(q, \pi/2, \pi/2) = \frac{1}{\pi} \int_0^{\pi} q(x)dx,$$

then $q(x) = \mu_0(q, \pi/2, \pi/2)$ *a.e. on* $(0, \pi)$.

Further, in [Yur13], Yurko proved the following generalization.

Theorem 1.3.4 (Yurko [Yur13]). *Let*

$$\mu_0(q) = \mu_0(q_0) + \frac{(\hat{q}\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)},$$

where $\varphi_0(x)$ is an eigenfunction of L_0 related to $\mu_0(q_0)$.

Then $q(x) = q_0(x) + \mu_0(q) - \mu_0(q_0)$ a.e. on $(0, \pi)$.

Our theorem with the lowest eigenvalue is as follows.

Theorem 1.3.5. *Let q, $q_0 \in L^1_{\mathbb{R}}(0, \pi)$. If*

$$\mu_0(q) - \mu_0(q_0) = \operatorname{ess\,inf} \hat{q} \quad \textit{or} \quad \mu_0(q) - \mu_0(q_0) = \operatorname{ess\,sup} \hat{q},$$

then $q(x) = q_0(x) + \mu_0(q) - \mu_0(q_0)$ a.e. on $(0, \pi)$.

Proof. Denote $(y, z) = \int_0^\pi y(x)\overline{z(x)}dx$, and $\varphi_0 := \varphi(x, \mu_0(q))$, $\varphi_0^0 = \varphi(x, \mu_0(q_0))$. One has

$$(\ell\varphi_0, \varphi_0^0) = \mu_0(q)(\varphi_0, \varphi_0^0),$$

$$(\ell_0\varphi_0, \varphi_0^0) = (\varphi_0, \ell_0\varphi_0^0) = \mu_0(q_0)(\varphi_0, \varphi_0^0),$$

and consequently,

$$((\hat{q} + \mu_0(q_0) - \mu_0(q))\varphi_0, \varphi_0^0) = 0.$$

By Sturm's oscillation theorem (see, e.g., [FY01, AHP13]), the product $\varphi_0\varphi_0^0$ has no zeros on interval $(0, \pi)$, hence $\hat{q} + \mu_0(q_0) - \mu_0(q) = 0$ a.e. on $(0, \pi)$. □

Remark 1.3.1. *Theorems 1.3.4 and 1.3.5 are true for Sturm-Liouville problems with arbitrary self-adjoint boundary conditions.*

From Theorem 1.3.5 for $\mu_0(q, \alpha, \beta)$ we get the following bounds.

Theorem 1.3.6. *Let $\alpha \in (0, \pi]$ and $\beta \in [0, \pi)$ and $q \in L^1_{\mathbb{R}}(0, \pi)$. The lowest eigenvalue $\mu_0(q, \alpha, \beta)$ has the property*

$$\operatorname{ess\,inf} q(x) + \mu_0(0, \alpha, \beta) \le \mu_0(q, \alpha, \beta) \le \operatorname{ess\,sup} q(x) + \mu_0(0, \alpha, \beta).$$

1.3.1.2 With one spectrum

Here, we give a new proof of the famous generalization of Ambarzumyan's theorem.

Theorem 1.3.7. *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$.

If $\mu_n(q,\alpha,\pi-\alpha) = \mu_n(0,\alpha,\pi-\alpha)$, *for all* $n \geq 0$, *then* $q(x) \equiv 0$.

We think that Theorem 1.3.7 is a natural generalization, because just one spectrum is used to reconstruct the potential q, without any additional conditions, as it is in the classical result.

Proof. Consider an operator $L(q,\alpha,\pi-\alpha)$ and an even operator[5] $L(0,\alpha,\pi-\alpha)$. N. Levinson proved [Lev49] (see also [Har09]), that an operator L is even if and only if

$$\varphi(\pi,\mu_n) = (-1)^n, \qquad n \geq 0. \tag{1.3.1}$$

The condition of the theorem means, that the operator $L(q,\alpha,\pi-\alpha)$ is isospectral with $L(0,\alpha,\pi-\alpha)$. Since the method of Jodeit and Levitan has described all the isospectral operators for potential function q, with $q' \in L^2(0,\pi)$, then the formulae (1.1.20)–(1.1.22) are held for operators $L(q,\alpha,\pi-\alpha)$ and $L(0,\alpha,\pi-\alpha)$. Therefore, taking into account, that $q_0(x) \equiv 0$, $\alpha_0 = \alpha$, $\beta_0 = \beta = \pi - \alpha$ and (1.3.1), then the relations (1.1.20)–(1.1.22), which connect these two operators, will become

$$q(x) = 2\frac{d}{dx}K(x,x), \tag{1.3.2}$$

$$\sum_{n=0}^{\infty} c_n = 0. \tag{1.3.3}$$

$$\sum_{n=0}^{\infty} \frac{c_n}{1 + c_n \tilde{a}_n^0} = 0. \tag{1.3.4}$$

If we subtract (1.3.3) from (1.3.4) we will obtain

$$\sum_{n=0}^{\infty} \frac{c_n^2 \tilde{a}_n^0}{1 + c_n \tilde{a}_n^0} = 0. \tag{1.3.5}$$

[5] A problem $L(q,\alpha,\beta)$ is said to be even, if $q(x) = q(\pi - x)$ and $\alpha + \beta = \pi$.

Since $1 + c_n \tilde{a}_n^0 > 0$ and $\tilde{a}_n^0 > 0$, for all $n \geq 0$, then from the equation (1.3.5) we obtain, that $c_n = 0,\ n \geq 0$. Thus, from the equations (1.1.18), (1.1.19) and (1.3.2) it follows that $q(x) \equiv 0$. □

Remark 1.3.2. *We will get the classical Ambarzumyan's theorem, if we take $\alpha = \pi/2$.*

1.3.2 On theorem of Marchenko

One of the main results of the present section is the following theorem, which, in some sense, is a generalization of Marchenko's uniqueness theorem.

Theorem 1.3.8. *Let $q' \in L^2_{\mathbb{R}}(0, \pi)$. If*

$$\mu_n(q, \alpha_0, \beta) = \mu_n(q_0, \alpha_0, \beta_0), \tag{1.3.6}$$

$$\tilde{a}_n(q, \alpha_0, \beta) \geq \tilde{a}_n(q_0, \alpha_0, \beta_0), \tag{1.3.7}$$

for all $n \geq 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

This kind of uniqueness theorem has not been considered before. The main difference between Theorems 1.3.2 and 1.3.8 is that the equality in Theorem 1.3.2 on norming constants we replace with inequality in (1.3.7). Note, we assume $q' \in L^2_{\mathbb{R}}(0, \pi)$ instead of general $q \in L^1_{\mathbb{R}}(0, \pi)$, since our proof is based on the results of Jodeit and Levitan (see [JL97] and Section 1.1). And the parameter α of boundary condition is in advance fixed $\alpha = \alpha_0$.

Proof. Consider operators $L_0 = L(q_0, \alpha_0, \beta_0)$ and $L = L(q, \alpha_0, \beta)$, with the set of norming constants $\tilde{a}_n^0 = \tilde{a}_n(q_0, \alpha_0, \beta_0)$ and $\tilde{a}_n = \tilde{a}_n(q, \alpha_0, \beta)$, $n \geq 0$, respectively. It is known (see, e.g. [JL97]), that in this case the kernel $F(x, y)$ of the integral equation (1.1.11) is

$$F(x, y) = \sum_{n=0}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0} \right) \varphi_0(x, \mu_n^0) \varphi_0(y, \mu_n^0). \tag{1.3.8}$$

Since by the condition of Theorem 1.3.8 the operators L and L_0 are isospectral, then the formulae (1.1.20)–(1.1.22) are held. If we compare the kernels (1.1.18) and (1.3.8), we'll refer, that $c_n = \dfrac{1}{\tilde{a}_n} - \dfrac{1}{\tilde{a}_n^0}$. So the formulae (1.1.21) and (1.1.22) will become

$$\cot\alpha = \cot\alpha_0 + \sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right), \tag{1.3.9}$$

$$\cot\beta = \cot\beta_0 + \sum_{n=0}^{\infty}(\tilde{a}_n^0 - \tilde{a}_n)\frac{\varphi_0^2(\pi,\mu_n^0)}{(\tilde{a}_n^0)^2}. \tag{1.3.10}$$

Thus, we have all the operators $L(q,\alpha,\beta)$ isospectral with $L(q_0,\alpha_0,\beta_0)$.

We supposed, that $\alpha = \alpha_0$, then by formula (1.3.9) we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{\tilde{a}_n} - \frac{1}{\tilde{a}_n^0}\right) = 0. \tag{1.3.11}$$

Since $\tilde{a}_n \geq \tilde{a}_n^0$, for all $n \geq 0$, thus from the equation (1.3.11) it refers that $\tilde{a}_n = \tilde{a}_n^0$, for all $n \geq 0$. Thus, from Marchenko uniqueness Theorem 1.3.2 we obtain $q(x) \equiv q_0(x)$ and $\beta = \beta_0$.

This completes the proof.

□

Remark 1.3.3. *Some analogues of Theorem 1.3.8 will be stated in Section 1.3.3.*

Remark 1.3.4. *From the relation* (1.3.10) *it follows, that we can assume $\beta = \beta_0$, instead of $\alpha = \alpha_0$. Even so, we will obtain $q(x) \equiv q_0(x)$ and $\alpha = \alpha_0$.*

1.3.3 Uniqueness theorems conditioned by inequalities

Consider two isospectral operators $L(q,\alpha,\beta)$ and $L(q_0,\alpha_0,\beta_0)$, with $\alpha,\alpha_0,\beta,\beta_0 \in (0,\pi)$ and $q' \in L^2_{\mathbb{R}}(0,\pi)$. Formulae, analogues to (1.1.21) and (1.1.22), can be obtained for κ_n:

$$\cot\alpha = \cot\alpha_0 + \sum_{n=0}^{\infty}\frac{1}{|\dot{\Phi}(\mu_n^0)|}\left(\frac{1}{|\kappa_n|} - \frac{1}{|\kappa_n^0|}\right), \tag{1.3.12}$$

$$\cot\beta = \cot\beta_0 + \sum_{n=0}^{\infty}\frac{|\kappa_n^0| - |\kappa_n|}{|\dot{\Phi}(\mu_n^0)|}. \tag{1.3.13}$$

In 2009, Harutyunyan proved a uniqueness theorem with the set of eigenvalues and the set of $\{\kappa_n\}_{n=0}^{\infty}$ (see (1.1.13)):

Theorem 1.3.9 (Harutyunyan [Har09]). *If*

$$\mu_n(q,\alpha,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$\kappa_n(q,\alpha,\beta) = \kappa_n(q_0,\alpha_0,\beta_0),$$

for all $n \geq 0$, *then* $q(x) = q_0(x)$ *a.e on* $(0,\pi)$ *and* $\alpha = \alpha_0$, $\beta = \beta_0$.

From Theorem 1.3.9 and formulae (1.1.12), (1.3.12), (1.3.13), new statement, similar to Theorem 1.3.8, can be proven for κ_n:

Theorem 1.3.10. *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$. *If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$|\kappa_n(q,\alpha_0,\beta)| \geq |\kappa_n(q_0,\alpha_0,\beta_0)|,$$

for all $n \geq 0$, *then* $\beta = \beta_0$ *and* $q(x) \equiv q_0(x)$.

Theorem 1.3.11. *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$. *If*

$$\mu_n(q,\alpha,\beta_0) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$|\varphi(\pi,\mu_n(q))| \leq |\varphi(\pi,\mu_n(q_0))|,$$

for all $n \geq 0$, *then* $\alpha = \alpha_0$ *and* $q(x) \equiv q_0(x)$.

Since the uniqueness theorem of Marchenko is also true for norming constants b_n, taking into consideration the relations (1.1.13), (1.1.15) and (1.3.14), analogues to Theorem 1.3.8 can be proven for $\psi(0,\mu_n)$ and b_n

Theorem 1.3.12. *Let* $q' \in L^2_{\mathbb{R}}(0,\pi)$. *If*

$$\mu_n(q,\alpha_0,\beta) = \mu_n(q_0,\alpha_0,\beta_0),$$

$$b_n(q,\alpha_0,\beta) \leq b_n(q_0,\alpha_0,\beta_0),$$

for all $n \geq 0$, *then* $\beta = \beta_0$ *and* $q(x) \equiv q_0(x)$.

There is a relationship between norming constants and characteristic functions:

$$b_n = \psi(0, \mu_n)\dot{\Psi}(\mu_n), \qquad n = 0, 1, 2, \dots . \tag{1.3.14}$$

Theorem 1.3.13. *Let $q' \in L^2_{\mathbb{R}}(0, \pi)$. If*

$$\mu_n(q, \alpha_0, \beta) = \mu_n(q_0, \alpha_0, \beta_0),$$

$$|\psi(0, \mu_n(q))| \leq |\psi(0, \mu_n(q_0))|,$$

for all $n \geq 0$, then $\beta = \beta_0$ and $q(x) \equiv q_0(x)$.

Remark 1.3.5. *In Theorems 1.3.10–1.3.13 instead of one boundary parameter we can fix the other. Even so, the results are valid.*

CHAPTER 2

Canonical Dirac system

2.1 Isospectral Dirac operators

2.1.1 Introduction

Let E is two dimensional identical matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

are well-known Pauli matrices, which have properties $\sigma_k^2 = E$, $\sigma_k^* = \sigma_k$ (self-adjointness) and $\sigma_k \sigma_j = -\sigma_j \sigma_k$ (anti-commutativity), when $k \neq j$, for $k, j = 1, 2, 3$.

Let p and q are real-valued, summable on $[0, \pi]$ functions, i.e. $p, q \in L^1_{\mathbb{R}}[0, \pi]$. By $L(p, q, \alpha, \beta) = L(\Omega, \alpha, \beta)$ we denote the boundary-value problem for canonical Dirac system (see [Tit61, GL66, LS70, Mar77, GD75]):

$$\ell y \equiv \left\{ B \frac{d}{dx} + \Omega(x) \right\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.1.1}$$

$$y_1(0) \cos \alpha + y_2(0) \sin \alpha = 0, \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right], \tag{2.1.2}$$

$$y_1(\pi) \cos \beta + y_2(\pi) \sin \beta = 0, \quad \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right], \tag{2.1.3}$$

where

$$B = \frac{1}{i} \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \sigma_2 p(x) + \sigma_3 q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

Matrix-function $\Omega(\cdot)$ is usually called potential function, and λ is a complex (spectral) parameter, $\lambda \in \mathbb{C}$.

By the same $L(p, q, \alpha, \beta)$ we also denote a self-adjoint operator, generated by differential expression ℓ in Hilbert space of two component vector-functions $L^2([0, \pi]; \mathbb{C}^2)$ (see, e.g. [LS88, Nai69]), on the domain

$$D = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in AC[0, \pi], (\ell y)_k \in L^2[0, \pi], k = 1, 2; \right.$$

$$\left. y_1(0) \cos\alpha + y_2(0) \sin\alpha = 0, \ y_1(\pi) \cos\beta + y_2(\pi) \sin\beta = 0 \right\}$$

where $AC[0, \pi]$ is the set of absolutely continuous functions on $[0, \pi]$. The scalar product in $L^2([a, b]; \mathbb{C}^2)$ we denote by $(f, g) = \int_a^b \langle f, g\rangle dx = \int_a^b [f_1(x)\bar{g}_1(x) + f_2(x)\bar{g}_2(x)]dx$.

Definition 2.1.1. *The values of the parameter λ, for which the system* (2.1.1) *has non trivial solutions from $D \subset L^2((-\infty, \infty); \mathbb{C}^2)$ are called eigenvalues and the corresponding solutions are called eigenfunctions of the operator $L(p, q)$.*

It is well known (see [GD75, HA06, AHM05]) that under these conditions the spectrum of the operator $L(p, q, \alpha, \beta)$ is purely discrete and consists of simple, real eigenvalues, which we denote by $\lambda_n = \lambda_n(p, q, \alpha, \beta) = \lambda_n(\Omega, \alpha, \beta)$, $n \in \mathbb{Z}$, to emphasize the dependence of λ_n on quantities p, q and α, β. It is also well known (see, e.g. [GD75, HA06, AHM05]) that the eigenvalues form a sequence, unbounded below as well as above. So we will enumerate it as $\lambda_k < \lambda_{k+1}, k \in \mathbb{Z}$, $\lambda_k > 0$, when $k > 0$ and $\lambda_k < 0$, when $k < 0$, and the nearest to zero eigenvalue we will denote by λ_0. If there are two eigenvalues nearest to zero, then by λ_0 we will denote the negative one. With this enumeration it is proved (see [GD75, HA06, AHM05]), that the eigenvalues have the asymptotics:

$$\lambda_n(\Omega, \alpha, \beta) = n + \frac{\beta - \alpha}{\pi} + r_n, \quad r_n = o(1), \quad n \to \pm\infty. \tag{2.1.4}$$

In what follows, writing $\Omega \in A$ will mean $p, q \in A$. If $\Omega \in L^2_{\mathbb{R}}[0, \pi]$, then we know, (see, e.g., [HA06]), that instead of $r_n = o(1)$ we have $\sum\limits_{n=-\infty}^{\infty} r_n^2 < \infty$.

Let $y(x,\lambda) = \varphi(x,\lambda,\alpha,\Omega)$ and $y(x,\lambda) = \psi(x,\lambda,\beta,\Omega)$ are the solutions of the Cauchy problems

$$\ell y = \lambda y, \qquad y(0,\lambda) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}, \tag{2.1.5}$$

$$\ell y = \lambda y, \qquad y(\pi,\lambda) = \begin{pmatrix} \sin\beta \\ -\cos\beta \end{pmatrix},$$

respectively. Since the differential expression ℓ is self-adjoint, the components $\varphi_1(x,\lambda)$, $\varphi_2(x,\lambda)$ and $\psi_1(x,\lambda)$, $\psi_2(x,\lambda)$ of the vector-functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ can be chosen real-valued for real λ. It is easy to see, that $\varphi_n(x,\Omega) = \varphi(x,\lambda_n,\alpha,\Omega)$ and $\psi_n(x,\Omega) = \psi(x,\lambda_n,\beta,\Omega)$ are the eigenfunctions, corresponding to the eigenvalue λ_n. By $a_n = a_n(\Omega,\alpha,\beta)$ and $b_n = b_n(\Omega,\alpha,\beta)$ we denote the squares of the L^2-norm of the eigenfunctions $\varphi_n(x,\Omega)$ and $\psi_n(x,\Omega)$:

$$a_n = \|\varphi_n\|^2 = \int_0^\pi |\varphi_n(x,\Omega)|^2 dx, \quad n \in \mathbb{Z},$$
$$b_n = \|\psi_n\|^2 = \int_0^\pi |\psi_n(x,\Omega)|^2 dx, \quad n \in \mathbb{Z}.$$

The numbers a_n and b_n are called norming constants. By $h_n(x,\Omega)$ we denote normalized eigenfunctions (i.e. $\|h_n(x)\| = 1$) of operator $L(\Omega,\alpha,\beta)$:

$$h_n(x) = h_n(x,\Omega) = \frac{\varphi_n(x,\Omega)}{\sqrt{a_n(\Omega,\alpha)}}, \tag{2.1.6}$$

and it can be taken also as

$$\hat{h}_n(x) = \hat{h}_n(x,\Omega) = \frac{\psi_n(x,\Omega)}{\sqrt{b_n(\Omega,\beta)}}.$$

It is easy to see, that $|h_n(0)|^2 = \dfrac{1}{a_n}$ and $|\hat{h}_n(\pi)|^2 = \dfrac{1}{b_n}$. It is known (see [GD75, HA06]) that in the case of $\Omega \in L^2_{\mathbb{R}}[0,\pi]$ the norming constants have the following asymptotic form:

$$a_n(\Omega) = \pi + c_n, \quad b_n(\Omega) = \pi + \tilde{c}_n, \quad \sum_{n=-\infty}^{\infty} c_n^2, \sum_{n=-\infty}^{\infty} \tilde{c}_n^2 < \infty, \tag{2.1.7}$$

2.1.2 Preliminary results

It is known, that in the case of Sturm-Liouville problem, with the help of the spectral function the potential q and boundary parameters α and β can be uniquely determined (see, e.g., [Mar52, GL64]). In case of Dirac operator it is not true (see [GL66]). If $\omega(x)$ is an absolutely continuous function, then by substituting $z(x,\lambda) = A(x)\varphi(x,\lambda,\alpha)$, where the unitary matrix

$$A(x) = \begin{pmatrix} \cos\omega(x) & \sin\omega(x) \\ -\sin\omega(x) & \cos\omega(x) \end{pmatrix},$$

the system (2.1.1) transforms to the system

$$\ell z \equiv \left\{ B\frac{d}{dx} + \tilde{\Omega}(x) \right\} z = \lambda z,$$

where $\tilde{\Omega}(x) = A^{-1}(x)BA'(x) + A^{-1}(x)\Omega(x)A(x)$, but the spectral function stays the same, i.e. the eigenvalues and norming constants stay the same. In order $\tilde{\Omega}(x)$ to be canonical, i.e. $\tilde{\Omega}(x)$ to have a form $\sigma_2\tilde{p}(x) + \sigma_3\tilde{q}(x)$, it is required that $\omega(x) = const = \omega_0$. Now the constant matrix

$$A(x) = \begin{pmatrix} \cos\omega_0 & \sin\omega_0 \\ -\sin\omega_0 & \cos\omega_0 \end{pmatrix},$$

transforms problem $L(\Omega,\alpha,\beta)$ to problem $L(A^{-1}\Omega A, \alpha-\omega_0, \beta-\omega_0)$, and they have the same spectral function. Here we can see, that if we fix one of the boundary parameters, we get $\omega_0 = 0$. So, to reconstruct the problem $L(\Omega,\alpha,\beta)$ uniquely with spectral function, we should fix one of the boundary parameters.

Therefore, in this section we consider Dirac operator $L(\Omega,\alpha,\beta)$ with one fixed boundary condition, e.g. $\beta = 0$, hence here we consider the problem $L(\Omega,\alpha,0)$.

Definition 2.1.2. *Two Dirac operators $L(\Omega,\alpha,0)$ and $L(\tilde{\Omega},\tilde{\alpha},0)$ are said to be isospectral, if $\lambda_n(\Omega,\alpha,0) = \lambda_n(\tilde{\Omega},\tilde{\alpha},0)$, for every $n \in \mathbb{Z}$.*

Lemma 2.1.1. *Let $\Omega, \tilde{\Omega} \in L^1_{\mathbb{R}}[0,\pi]$ and the operators $L(\Omega,\alpha,0)$ and $L(\tilde{\Omega},\tilde{\alpha},0)$ are isospectral. Then $\tilde{\alpha} = \alpha$.*

Proof. The proof follows from the asymptotics (2.1.4):

$$\frac{\alpha}{\pi} = \lim_{n\to\infty} (n - \lambda_n(\Omega, \alpha, 0)) = \lim_{n\to\infty} (n - \lambda_n(\tilde{\Omega}, \tilde{\alpha}, 0)) = \frac{\tilde{\alpha}}{\pi}.$$

□

So, instead of isospectral operators $L(\Omega, \alpha, 0)$ and $L(\tilde{\Omega}, \tilde{\alpha}, 0)$, we can talk about "isospectral potentials" Ω and $\tilde{\Omega}$.

Theorem 2.1.1. *(Uniqueness theorem). The map*

$$(\Omega, \alpha) \in L^2_{\mathbb{R}}[0, \pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \longleftrightarrow \{\lambda_n(\Omega, \alpha, 0), a_n(\Omega, \alpha, 0); n \in \mathbb{Z}\}$$

is one-to-one.

Remark 2.1.1. *It is natural to call this a Marchenko Theorem, since it is an analogue of the famous theorem of V.A. Marchenko [Mar50, Mar52], in the case for Sturm-Liouville problem. The proof of this theorem for the case $p, q \in AC[0, \pi]$ is in the paper [Wat99]. The detailed proof for the case $p, q \in L^2_{\mathbb{R}}[0, \pi]$ is in [Har10b] (see also [GL66, GD75, Har94, Hor01, FY01, WW15]).*

Let us fix some $\Omega \in L^2_{\mathbb{R}}[0, \pi]$ and consider the set of all canonical potentials $\tilde{\Omega} = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & -\tilde{p} \end{pmatrix}$, with the same spectrum as Ω:

$$M^2(\Omega) = \{\tilde{\Omega} \in L^2_{\mathbb{R}}[0, \pi] : \lambda_n(\tilde{\Omega}, \tilde{\alpha}, 0) = \lambda_n(\Omega, \alpha, 0), n \in \mathbb{Z}\}.$$

Our main goal is to give the description of the set $M^2(\Omega)$ as explicit as it possible.

From the uniqueness theorem it is easily follows:

Corollary 2.1.1. *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{a_n(\tilde{\Omega}), n \in \mathbb{Z}\}$$

is one-to-one.

Since $\tilde{\Omega} \in M^2(\Omega)$, then $a_n(\tilde{\Omega})$ have similar to (2.1.7) asymptotics. Since $a_n(\Omega)$ and $a_n(\tilde{\Omega})$ are positive numbers, there exist real numbers $t_n = t_n(\tilde{\Omega})$, such that $\dfrac{a_n(\Omega)}{a_n(\tilde{\Omega})} = e^{t_n}$. From the latter equality and from (2.1.7) follows that

$$e^{t_n} = 1 + d_n, \qquad \sum_{n=-\infty}^{\infty} d_n^2 < \infty.$$

It is easy to see, that the sequence $\{t_n; n \in \mathbb{Z}\}$ is also from l^2, i.e. $\sum_{n=-\infty}^{\infty} t_n^2 < \infty$. Since all $a_n(\Omega)$ are fixed, then from the corollary 2.1.1 and the equality $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}$ we will get:

Corollary 2.1.2. *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{t_n(\tilde{\Omega}), n \in \mathbb{Z}\} \in l^2$$

is one-to-one.

Thus, each isospectral potential is uniquely determined by a sequence $\{t_n; n \in \mathbb{Z}\}$. Note, that the problem of description of isospectral Sturm-Liouville operators was solved in [IT83, IMT84, DT84, PT87, KC09, JL97].

For Dirac operators the description of $M^2(\Omega)$ is given in [Har94]. That description has a "recurrent" form, i.e. at the first in [Har94] is given the description of a family of isospectral potentials $\Omega(x,t), t \in \mathbb{R}$, for which only one norming constant $a_m(\Omega(\cdot,t))$ is different from $a_m(\Omega)$ (namely, $a_m(\Omega(\cdot,t)) = a_m(\Omega)e^{-t}$), while the others are equal, i.e. $a_m(\Omega(\cdot,t)) = a_m(\Omega)$, when $n \neq m$.

Theorem 2.1.2. *[Har94]. Let $t \in \mathbb{R}$, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and* [1]

$$\Omega(x,t) = \Omega(x) + \frac{e^t - 1}{\theta_m(x,t,\Omega)}\{Bh_m(x,\Omega)h_m^*(x,\Omega) - h_m(x,\Omega)h_m^*(x,\Omega)B\},$$

where $\theta_n(x,t,\Omega) = 1 + (e^t - 1)\int_0^x |h_n(s,\Omega)|^2 ds$. Then, for arbitrary $t \in \mathbb{R}$, $\lambda_n(\Omega,t) = \lambda_n(\Omega)$ for all $n \in \mathbb{Z}$, $a_n(\Omega,t) = a_n(\Omega)$ for all $n \in \mathbb{Z}\backslash\{m\}$ and $a_m(\Omega,t) = a_m(\Omega)e^{-t}$. The normalized

[1]Here * is a sign of transponation, e.g. $h_m^* = \begin{pmatrix} h_{m_1} \\ h_{m_2} \end{pmatrix}^* = (h_{m_1}, h_{m_2})$

eigenfunctions of the problem $L(\Omega(\cdot,t),\alpha)$ *are given by the formulae:*

$$h_n(x,\Omega(\cdot,t)) = \begin{cases} \dfrac{e^{-t/2}}{\theta_m(x,t,\Omega)} h_m(x,\Omega), & if\ n = m \\ h_n(x,\Omega) - \dfrac{(e^t-1)\int_0^x h_m^*(s,\Omega)h_n(s,\Omega)ds}{\theta_m(x,t,\Omega)} h_m(x,\Omega), & if\ n \neq m; \end{cases}$$

Theorem 2.1.2 shows that it is possible to change exactly one norming constant, keeping the others. As examples of isospectral potentials Ω and $\tilde{\Omega}$ we can present

$$\Omega(x) \equiv 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{\Omega}(x) = \Omega_{m,t}(x) = \frac{\pi(e^t-1)}{\pi + (e^t-1)x} \begin{pmatrix} -\sin 2mx & \cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix},$$

where $t \in \mathbb{R}$ is an arbitrary real number and $m \in \mathbb{Z}$ is an arbitrary integer.

Changing successively each $a_m(\Omega)$ by $a_m(\Omega)e^{-t_m}$, we can obtain any isospectral potential, corresponding to the sequence $\{t_m; m \in \mathbb{Z}\} \in l^2$. It follows from the uniqueness Theorem 2.1.1 that the sequence, in which we change the norming constants, is not important.

In [Har94] were used the following designations:

$$\begin{aligned}
&T_{-1} = \{\ldots,0,\ldots\},\\
&T_0 = \{\ldots,0,\ldots,0,t_0,0,\ldots,0,\ldots\},\\
&T_1 = \{\ldots,0,\ldots,0,0,t_0,t_1,0,\ldots,0,\ldots\},\\
&T_2 = \{\ldots,0,\ldots,0,t_{-1},t_0,t_1,0,\ldots,0,\ldots\},\\
&\ldots,\\
&T_{2n} = \{\ldots,0,0,t_{-n},\ldots,t_{-1},t_0,t_1,\ldots,t_{n-1},t_n,0,\ldots\},\\
&T_{2n+1} = \{\ldots,0,t_{-n},t_{-n+1},\ldots,t_{-1},t_0,t_1,\ldots,t_n,t_{n+1},0,\ldots\},\\
&\ldots.
\end{aligned}$$

Let $\Omega(x,T_{-1}) \equiv \Omega(x)$ and

$$\Omega(x,T_m) = \Omega(x,T_{m-1}) + \triangle\Omega(x,T_m), \quad m = 0,1,2,\ldots,$$

where

$$\triangle\Omega(x,T_m) = \frac{e^{t_{\tilde{m}}}-1}{\theta_m(x,t_{\tilde{m}},\Omega(\cdot,T_{m-1}))}[Bh_{\tilde{m}}(x,\Omega(\cdot,T_{m-1}))h^*_{\tilde{m}}(\cdot) - h_{\tilde{m}}(\cdot)h^*_{\tilde{m}}(\cdot)B],$$

where $\tilde{m} = \frac{m+1}{2}$, if m is odd and $\tilde{m} = -\frac{m}{2}$, if m is even. The arguments in others $h_{\tilde{m}}(\cdot)$ and $h^*_{\tilde{m}}(\cdot)$ are the same as in the first. And after that in [Har94] was proved:

Theorem 2.1.3. *[Har94]. Let $T = \{t_n, n\in\mathbb{Z}\} \in l^2$ and $\Omega \in L^2_{\mathbb{R}}[0,\pi]$. Then*

$$\Omega(x,T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \triangle\Omega(x,T_m) \in M^2(\Omega). \tag{2.1.8}$$

We see, that each potential matrix $\triangle\Omega(x,T_m)$ defined by normalized eigenfunctions $h_{\tilde{m}}(x,\Omega(x,T_{m-1}))$ of the previous operator $L(\Omega(\cdot,T_{m-1}),\alpha)$. This approach we call "recurrent" description.

2.1.3 Isospectral operators

We want to give a description of the set $M^2(\Omega)$ only in terms of eigenfunctions $h_n(x,\Omega)$ of the initial operator $L(\Omega,\alpha,0)$ and sequence $T\in l^2$. With this aim, let us denote by $N(T_m)$ the set of the positions of the numbers in T_m, which are not necessary zero, i.e.

$N(T_0) = \{0\}$,

$N(T_1) = \{0,1\}$,

$N(T_2) = \{-1,0,1\}$,

$\ldots,$

$N(T_{2n}) = \{-n,-(n-1),\ldots,0,\ldots,n-1,n\}$,

$N(T_{2n+1}) = \{-n,-(n-1),\ldots,0,\ldots,n,n+1\}$,

$\ldots,$

in particular $N(T)\equiv\mathbb{Z}$. By $S(x,T_m)$ we denote $(m+1)\times(m+1)$ square matrix

$$S(x,T_m) = \left(\delta_{ij} + (e^{t_j}-1)\int_0^x h_i^*(s)h_j(s)ds\right)_{i,j\in N(T_m)}$$

where δ_{ij} is a Kronecker symbol. By $S_p^{(k)}(x,T_m)$ we denote a matrix, which is obtained from the matrix $S(x,T_m)$ when we replace the $k-th$ column of $S(x,T_m)$ by $H_p(x,T_m) = \{-(e^{t_k}-1)h_{k_p}(x)\}_{k\in N(T_m)}$ column, $p=1,2$. Now we can formulate our result as follow:

Theorem 2.1.4. *Let $T=\{t_k\}_{k\in\mathbb{Z}} \in l^2$ and $\Omega \in L^2_{\mathbb{R}}[0,\pi]$. Then the isospectral potential from $M^2(\Omega)$, corresponding to T, is given by formula*

$$\Omega(x,T) = \Omega(x) + G(x,x,T)B - BG(x,x,T) = \begin{pmatrix} p(x,T) & q(x,T) \\ q(x,T) & -p(x,T) \end{pmatrix},$$

where

$$G(x,x,T) = \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \begin{pmatrix} \det S_1^{(k)}(x,T) \\ \det S_2^{(k)}(x,T) \end{pmatrix} h_k^*(x),$$

and $\det S(x,T) = \lim\limits_{m\to\infty} \det S(x,T_m)$ *(the same for* $\det S_p^k(x,T), p=1,2$*).*

In addition, for $p(x,T)$ and $q(x,T)$ we get an explicit representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} \det S_p^{(k)}(x,T) h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)} \sum_{k\in\mathbb{Z}} \sum_{p=1}^{2} (-1)^{p-1} \det S_p^{(k)}(x,T) h_{k_p}(x).$$

Proof. The spectral function of an operator $L(\Omega,\alpha,0)$ is defined as

$$\rho(\lambda) = \begin{cases} \displaystyle\sum_{0<\lambda_n\le\lambda} \frac{1}{a_n(\Omega)}, & \lambda > 0, \\ \displaystyle -\sum_{\lambda<\lambda_n\le 0} \frac{1}{a_n(\Omega)}, & \lambda < 0, \end{cases}$$

i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda=\lambda_n$ equals $\dfrac{1}{a_n}$ and $\rho(0)=0$.

Let $\Omega, \tilde{\Omega} \in L^2_{\mathbb{R}}[0,\pi]$ and they are isospectral. It is known (see [GL66, LS88, AHM05, Har08]), that there exists a function $G(x,y)$ such that:

$$\varphi(x,\lambda,\alpha,\tilde{\Omega}) = \varphi(x,\lambda,\alpha,\Omega) + \int_0^x G(x,s)\varphi(s,\lambda,\alpha,\Omega)dt.$$

It is also known (see, e.g. [GL66, LS88, AHM05]), that the function $G(x, y)$ satisfies to the Gel'fand-Levitan integral equation:

$$G(x, y) + F(x, y) + \int_0^x G(x, s)F(s, y)ds = 0, \quad 0 \leq y \leq x, \tag{2.1.9}$$

where

$$F(x, y) = \int_{-\infty}^{\infty} \varphi(x, \lambda, \alpha, \Omega)\varphi^*(y, \lambda, \alpha, \Omega)d[\tilde{\rho}(\lambda) - \rho(\lambda)],$$

If the potential $\tilde{\Omega}$ from $M^2(\Omega)$ is such that only finite norming constants of the operator $L(\tilde{\Omega}, \alpha, 0)$ are different from the norming constants of the operator $L(\Omega, \alpha, 0)$, i.e. $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}, n \in N(T_m)$ and the others are equal, then we have

$$d\tilde{\rho}(\lambda) - d\rho(\lambda) = \sum_{k\in N(T_m)} \Big(\frac{1}{\tilde{a}_k} - \frac{1}{a_k}\Big)\delta(\lambda - \lambda_k)d\lambda = \sum_{k\in N(T_m)} \Big(\frac{e^{t_k} - 1}{a_k}\Big)\delta(\lambda - \lambda_k)d\lambda,$$

where δ is Dirac δ-function. In this case the kernel $F(x, y)$ can be written in a form of a finite sum (using notation (2.1.6)):

$$F(x, y) = F(x, y, T_m) = \sum_{k\in N(T_m)} (e^{t_k} - 1)h_k(x, \Omega)h_k^*(y, \Omega), \tag{2.1.10}$$

and consequently, the integral equation (2.1.9) becomes to an integral equation with degenerated kernel, i.e. it becomes to a system of linear equations and we look for the solution in the following form:

$$G(x, y, T_m) = \sum_{k\in N(T_m)} g_k(x)h_k^*(y), \tag{2.1.11}$$

where $g_k(x) = \begin{pmatrix} g_{k_1}(x) \\ g_{k_2}(x) \end{pmatrix}$ is unknown vector-function. Putting the expressions (2.1.10) and (2.1.11) into the integral equation (2.1.9) we will obtain a system of algebraic equations for determining the functions $g_k(x)$:

$$g_k(x) + \sum_{i\in N(T_m)} s_{ik}(x)g_i(x) = -(e^{t_k} - 1)h_k(x), \quad k \in N(T_m), \tag{2.1.12}$$

where

$$s_{ik}(x) = (e^{t_k} - 1)\int_0^x h_i^*(s)h_k(s)ds.$$

It would be better if we consider the equations (2.1.12) for the vectors $g_k = \begin{pmatrix} g_{k_1} \\ g_{k_2} \end{pmatrix}$ by coordinates g_{k_1} and g_{k_2} to be a system of scalar linear equations:

$$g_{k_p}(x) + \sum_{i \in N(T_m)} s_{ik}(x) g_{i_p}(x) = -(e^{t_k} - 1) h_{k_p}(x), \quad k \in N(T_m), \quad p = 1, 2. \tag{2.1.13}$$

The systems (2.1.13) might be written in matrix form

$$S(x, T_m) g_p(x, T_m) = H_p(x, T_m), \qquad p = 1, 2,$$

where the column vectors $g_p(x, T_m) = \{g_{k_p}(x, T_m)\}_{k \in N(T_m)}$, $p = 1, 2$, and the solution can be found in the form (Cramer's rule):

$$g_{k_p}(x, T_m) = \frac{\det S_p^{(k)}(x, T_m)}{\det S(x, T_m)}, \quad k \in N(T_m), \quad p = 1, 2.$$

Thus we have obtained for $g_k(x)$ the following representation:

$$g_k(x, T_m) = \frac{1}{\det S(x, T_m)} \begin{pmatrix} \det S_1^{(k)}(x, T_m) \\ \det S_2^{(k)}(x, T_m) \end{pmatrix} \tag{2.1.14}$$

and then by putting (2.1.14) into (2.1.11) we find the function $G(x, y, T_m)$. If the potential Ω is from $L^1_{\mathbb{R}}$, then such is also the kernel $G(x, x, T_m)$ (see [Har94]), and the relation between them gives as follow:

$$\Omega(x, T_m) = \Omega(x) + G(x, x, T_m)B - BG(x, x, T_m). \tag{2.1.15}$$

On the other hand we have

$$\Omega(x, T_m) = \Omega(x) + \sum_{k=0}^{m} \triangle\Omega(x, T_k). \tag{2.1.16}$$

So, using the Theorem (2.1.3) and the equality (2.1.16) we can pass to the limit in (2.1.15), when $m \to \infty$:

$$\Omega(x, T) = \Omega(x) + G(x, x, T)B - BG(x, x, T). \tag{2.1.17}$$

The potentials Ω in (2.1.8) and (2.1.17) have the same spectral data $\{\lambda_n(T), a_n(T)\}_{n \in \mathbb{Z}}$, and therefore they are the same and $\Omega(\cdot, T)$ defined by (2.1.17) is also from $M^2(\Omega)$.

Using (2.1.11) and (2.1.14) we calculate the expression $G(x,x,T_m)B - BG(x,x,T_m)$ and pass to the limit, obtaining for the $p(x,T)$ and $q(x,T)$ the representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)} \sum_{k\in N(T)} \sum_{p=1}^{2} \det S_p^{(k)}(x,T) h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)} \sum_{k\in N(T)} \sum_{p=1}^{2} (-1)^{p-1} \det S_p^{(k)}(x,T) h_{k_p}(x).$$

Theorem is proved. □

For example, when we change just one norming constant (e.g. for T_0) we get two independent linear equations:

$$(1 + s_{00}(x)) g_{0_1}(x) = -(e^{t_0} - 1) h_{0_1}(x),$$

$$(1 + s_{00}(x)) g_{0_2}(x) = -(e^{t_0} - 1) h_{0_2}(x).$$

For function g we find

$$g_{0_1}(x) = -\frac{(e^{t_0} - 1) h_{0_1}(x)}{1 + s_{00}(x)},$$

$$g_{0_2}(x) = -\frac{(e^{t_0} - 1) h_{0_2}(x)}{1 + s_{00}(x)},$$

and for the potentials $p(x,T_0)$ and $q(x,T_0)$ we get the following representation

$$p(x,T_0) = p(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (2 h_{0_1}(x) h_{0_2}(x)),$$

$$q(x,T_0) = q(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)} (h_{0_2}^2(x) - h_{0_1}^2(x)).$$

2.2 Dirac operators with linear potential and its perturbations

In this section we consider Dirac operator with linear potential function on whole and half axes, and find the eigenvalues and eigenfunctions in explicit form. After we perturb spectral function and construct operators which are generated by that spectral function.

2.2.1 Operator on whole axis

Let p and q are real-valued, local integrable on $(-\infty, \infty)$ functions ($p, q \in L^1_{\mathbb{R},loc}(-\infty, \infty)$). By $L(p, q)$ we denote a self-adjoint operator (see [Nai69]), generated by differential expression ℓ (see (2.1.1)) in Hilbert space of two-component vector-functions $L^2((-\infty, \infty); \mathbb{C}^2)$ on the domain

$$D = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(-\infty, \infty) \cap AC(-\infty, \infty);$$
$$(\ell y)_k \in L^2(-\infty, \infty), k = 1, 2 \Big\}, \tag{2.2.1}$$

where $AC(-\infty, \infty) = AC(\mathbb{R})$ is the set of functions, which are absolutely continuous on each finite segment $[a, b] \subset (-\infty, \infty), -\infty < a < b < \infty$. We assume, that the spectrum of this operator is purely discrete (see, e.g. [Mar68, AH16]), and consists of simple eigenvalues, which we denote by $\lambda_n(p, q)$, $n \in \mathbb{Z}$.

At first we consider an operator $L(0, x)$ (with $p(x) \equiv 0$ and $q(x) \equiv x$), which corresponds to the system

$$\ell y \equiv \Big\{ B \frac{d}{dx} + \Omega_0(x) \Big\} y = \lambda y, \tag{2.2.2}$$

where $\Omega_0(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$, on the domain (2.2.1). This operator we call Dirac operator with linear potential.

As it follows from the results of [Mar68] and [LS70] the spectra of this operator is pure discrete and consists of simple eigenvalues. The eigenvalues of the operator $L(p, q)$ we will denote $\lambda_n(p,q)$ with corresponding enumeration. One of the sufficient conditions for discreteness of the spectra is (see [Mar68])

$$\lim_{|x|\to\infty}\left[p^2(x)+q^2(x)-\sqrt{(p'(x))^2+(q'(x))^2}\right]=\infty. \tag{2.2.3}$$

It is easy to see that in our case ($p(x)\equiv 0, q(x)\equiv x$) the condition (2.2.3) holds.

Writing the system (2.2.2) componentwise, we get

$$-y_1'+xy_1=\lambda y_2, \tag{2.2.4}$$

$$y_2'+xy_2=\lambda y_1, \tag{2.2.5}$$

we can obtain two second order differential equations for both y_1 and y_2 separately:

$$-y_1''+x^2y_1=(\lambda^2-1)y_1, \tag{2.2.6}$$

$$-y_2''+x^2y_2=(\lambda^2+1)y_2. \tag{2.2.7}$$

It is well known (see, e.g. [LS70]) that the equation

$$-y''+x^2y=\mu y$$

has solution from $L^2(-\infty,\infty)$ only for $\mu = 2n+1,\ n = 0,1,2,\dots,$ and corresponding solutions are Chebyshev-Hermite polynomials

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}. \tag{2.2.8}$$

Therefore, λ can be an eigenvalue of $L(0,x)$ only if $\lambda^2-1=2n+1$, i.e. $\lambda^2=2(n+1)$. Thus, if $\lambda=\lambda_{\pm n}=\pm\sqrt{2(n+1)}$, then the solutions of the equation (2.2.6) are

$$y_1(x)=H_n(x), \qquad n=0,1,2\dots.$$

At the same time $\lambda^2+1=2n+3=2(n+1)+1$ and consequently the solutions of the equation (2.2.7) are

$$y_2(x)=H_{n+1}(x), \qquad n=0,1,2\dots.$$

The Chebyshev-Hermite polynomials have the properties (see [LS70])

$$H'_n(x) = 2nH_{n-1}(x), \quad H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \qquad n = 1, 2, \ldots.$$

The general formulae for $H_n(x)$ are

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \cdots,$$

in which the last member is $(-1)^{\frac{n}{2}}\frac{n!}{(n/2)!}$, for even n and $(-1)^{\frac{(n-1)}{2}}\frac{n!}{((n-1)/2)!}2x$, for odd n. Thus, we note that $H_{2k+1}(0) = 0$, for $k = 0, 1, 2, \ldots$.

It is well known (see, e.g. [LS70]) that the squares of the L^2-norm of $H_n(x)$ with the weight e^{-x^2} is equal

$$\int_{-\infty}^{\infty} H_n^2(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}$$

and

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0, \qquad n, m = 0, 1, 2, \ldots, \quad n \neq m.$$

Therefore, if we take

$$\varphi_n(x) = C_n e^{-\frac{x^2}{2}} H_n(x), \qquad n = 0, 1, 2, \ldots, \tag{2.2.9}$$

where

$$C_n = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}, \tag{2.2.10}$$

then the system $\{\varphi_n(x)\}_{n=0}^{\infty}$ will became orthonormal system on whole real axis. It is called the system of Chebyshev-Hermite orthonormal functions. Now let us show that vector-functions

$$U_{-n}(x) = \begin{pmatrix} -\varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix},$$

$$U_0(x) = \begin{pmatrix} 0 \\ \varphi_0(x) \end{pmatrix}, \tag{2.2.11}$$

$$U_n(x) = \begin{pmatrix} \varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix},$$

for $n = 1, 2, \ldots$, corresponding to eigenvalues $\lambda_{-n} = -\sqrt{2n}, \lambda_0 = 0, \lambda_n = \sqrt{2n}$, are eigenfunctions of the operator $L(0, x)$. At first we will show that for $n = 1, 2, \ldots$:

$$\begin{cases} \varphi_n'(x) + x\varphi_n(x) = \sqrt{2n}\varphi_{n-1}(x), \\ -\varphi_{n-1}'(x) + x\varphi_{n-1}(x) = \sqrt{2n}\varphi_n(x). \end{cases} \tag{2.2.12}$$

Indeed, for $\varphi_n'(x), n = 1, 2, \ldots$, we have

$$\varphi_n'(x) = -C_n x e^{-\frac{x^2}{2}} H_n(x) + C_n e^{-\frac{x^2}{2}} H_n'(x) = C_n e^{-\frac{x^2}{2}} \left(H_n'(x) - xH_n(x)\right),$$

Putting these into the left side of the equation (2.2.5) and using the property $H'_n(x) = 2nH_{n-1}(x)$ we will get equalities

$$\varphi_n'(x) + x\varphi_n(x) =$$
$$C_n e^{-\frac{x^2}{2}} \left(H_n'(x) - xH_n(x)\right) + xC_n e^{-\frac{x^2}{2}} H_n(x) =$$
$$C_n e^{-\frac{x^2}{2}} H_n'(x) = C_n e^{-\frac{x^2}{2}} 2nH_{n-1}(x) =$$
$$\frac{C_n}{C_{n-1}} 2nC_{n-1} e^{-\frac{x^2}{2}} H_{n-1}(x) = \frac{2nC_n}{C_{n-1}} \varphi_{n-1}(x).$$

Taking into account (2.2.10), we see that the fraction $\dfrac{2nC_n}{C_{n-1}} = \sqrt{2n}$. Thus, we have

$$\varphi_n'(x) + x\varphi_n(x) = \sqrt{2n}\varphi_{n-1}(x), \qquad n = 1, 2, \ldots.$$

In the similar way we obtain the following equations (here we use the property $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$)

$$-\varphi_{n-1}'(x) + x\varphi_{n-1}(x) = \sqrt{2n}\varphi_n(x), \qquad n = 1, 2, \ldots.$$

Thus, we have $\ell U_n(x) = \sqrt{2n} U_n(x),\ n = 1, 2, \ldots$, i.e. $U_n(x),\ n = 1, 2, \ldots$, are the eigenfunctions of the operator $L(0, x)$ with the eigenvalues $\lambda_n(0, x) = \sqrt{2n},\ n = 1, 2, \ldots$.

$U_{-n}(x)$ will satisfy to the system

$$\begin{cases} \varphi_n'(x) + x\varphi_n(x) = (-\sqrt{2n})(-\varphi_{n-1}(x)), \\ -(-\varphi_{n-1}'(x)) + x(-\varphi_{n-1}(x)) = (-\sqrt{2n})\varphi_n(x). \end{cases} \tag{2.2.13}$$

In fact the systems (2.2.13) and (2.2.12) coincide, which means that for $n = 1, 2, \ldots$ $U_{-n}(x)$ are also the solutions (eigenfunctions) for the system (2.2.12) ((2.2.13)) with the eigenvalues $\lambda_{-n}(0, x) = -\sqrt{2n},\ n = 1, 2, \ldots$.

$U_0(x)$ satisfies to the system (2.2.2), when $\lambda_0(0, x) = 0$ (note that $\varphi_0(x) = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{x^2}{2}}$ and $\varphi_0'(x) = \frac{1}{\pi^{\frac{1}{4}}}(-x)e^{-\frac{x^2}{2}}$):

$$\begin{cases} \varphi_0'(x) + x\varphi_0(x) = \frac{1}{\pi^{\frac{1}{4}}}(-x)e^{-\frac{x^2}{2}} + x\frac{1}{\pi^{\frac{1}{4}}}e^{-\frac{x^2}{2}} = 0, \\ -0 + x \cdot 0 = 0 \cdot \frac{1}{\pi^{\frac{1}{4}}}e^{-\frac{x^2}{2}}. \end{cases}$$

So, such defined vector-functions $U_n(x),\ n \in \mathbb{Z}$ are eigenfunctions of the operator $L(0, x)$ with the eigenvalues $\lambda_n(0, x) = sign(n)\sqrt{2|n|},\ n \in \mathbb{Z}$.

2.2.2 Operators on half axis

Let us consider also canonical Dirac system on half axis. Let p and q are real-valued, local summable on $(0, \infty)$ functions, i.e. $p, q \in L^1_{\mathbb{R},loc}(0, \infty)$. For $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by $L(p, q, \alpha)$ we denote the self-adjoint operator, generated by differential expression ℓ (see (2.1.1)) in Hilbert space of two component vector-functions $L^2((0, \infty); \mathbb{C}^2)$ on the domain

$$D_\alpha = \Big\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; y_k \in L^2(0, \infty) \cap AC(0, \infty);$$

$$(\ell y)_k \in L^2(0, \infty), k = 1, 2;\ y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0 \Big\}$$

where $AC(0, \infty)$ is the set of functions, which are absolutely continuous on each finite segment $[a, b] \subset (0, \infty), 0 < a < b < \infty$. We assume, that the spectrum of this operator is purely discrete (see, e.g. [Mar68, AH16]), and consists of simple eigenvalues, which we denote by $\lambda_n(p, q, \alpha)$, $n \in \mathbb{Z}$. It is easy to see that if in boundary condition $y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0$ we take $\alpha = 0$, then we have condition

$$y_1(0) = 0, \tag{2.2.14}$$

and if we take $\alpha = \frac{\pi}{2}$, we obtain boundary condition

$$y_2(0) = 0. \tag{2.2.15}$$

Let $y = \varphi(x, \lambda, \alpha, \Omega)$ is the same as in the case of finite interval, i.e. $\varphi(x, \lambda)$ is the solution of Cauchy problem (2.1.5). Then $\varphi_n(x) = \varphi(x, \lambda_n)$ are the eigenfunctions, $a_n = \int_0^\infty |\varphi_n(x, \Omega)|^2 dx$, $n \in \mathbb{Z}$, are the norming constants, and $h_n(x) = h_n(x, \Omega, \lambda_n) = \dfrac{\varphi_n(x)}{\sqrt{a_n}}$ are the normalized eigenfunctions.

It is easy to see from (2.2.8)-(2.2.11) that the eigenfunctions of the operator $L(0, x, 0)$ are vector-functions $U_{2k}(x)$, which correspond to the eigenvalues $\lambda_k(0, x, 0) = \lambda_{2k}(0, x) = 2sign(k)\sqrt{|k|}$, $k \in \mathbb{Z}$. And the eigenfunctions of the operator $L(0, x, \pi/2)$ are vector-functions $U_{2k+1}(x)$ corresponding to the eigenvalue $\lambda_k(0, x, \pi/2) = \lambda_{2k+1}(0, x) = sign(2k+1)\sqrt{2|2k+1|}$, $k \in \mathbb{Z}$.

By $\psi = \psi(x, \lambda, \alpha, \Omega)$ we denote the solution of the Cauchy problem ($\alpha \in \mathbb{C}$)

$$\ell y = \lambda y, \qquad y(0) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix},$$

on $(0, \infty)$, and we denote this problem by $S(p, q, \lambda, \alpha)$. Such solution exists and unique and its components ψ_1 and ψ_2 are entire functions in parameters λ and α (see, e.g. [Har04]).

If $\alpha = 0$ and $\Omega = \Omega_0$, then $\psi(x, \lambda, 0, \Omega_0)$ satisfies to the boundary condition (2.2.14) and in order to be an eigenfunction of the operator $L(0, x, 0)$ it must be from $L^2(0, \infty; \mathbb{C}^2)$. As we have seen recently, it is possible only when $\lambda = \lambda_k(0, x, 0) = \lambda_{2k}(0, x) = 2sign(k)\sqrt{|k|}$, $k \in \mathbb{Z}$. Thus the eigenvalues and eigenfunctions of the operator $L(0, x, 0)$ are $\lambda_k(0, x, 0)$ and $\psi(x, \lambda_k(0, x, 0), 0, \Omega_0) = \psi(x, \lambda_{2k}(0, x), 0, \Omega_0)$, for $k \in \mathbb{Z}$.

Let us now consider Cauchy problems $S(0, x, \lambda_n(0, x, 0), 0)$, for $n \in \mathbb{Z}$. It is easy to see that the functions

$$V_n(x) = -\frac{U_{2n}(x)}{\varphi_{2n}(0)}, \qquad n \in \mathbb{Z}, \tag{2.2.16}$$

are the solutions of the these Cauchy problems. At the same time $V_n(x)$ are eigenfunctions of the operator $L(0, x, 0)$ which correspond to the eigenvalues $\lambda_n(0, x, 0)$, for $n \in \mathbb{Z}$. Since the solution of Cauchy problem is unique, it follows that

$$V_n(x) \equiv \psi(x, \lambda_n(0, x, 0), 0, \Omega_0), \qquad n \in \mathbb{Z}. \tag{2.2.17}$$

The squares of the L^2-norms of these functions

$$a_n = a_n(0,x) = \|V_n(\cdot)\|^2 = \int_0^\infty |V_{n,1}(x)|^2 + |V_{n,2}(x)|^2 dx$$

are called norming constants. Using (2.2.9)-(2.2.11) and (2.2.16) we can easily calculate the values of the norming constants:

$$a_0 = \frac{\pi^{1/2}}{2}, \quad a_{-n} = a_n = \frac{1}{|\varphi_{2n}(0)|^2} = \frac{4^n (n!)^2 \pi^{1/2}}{(2n)!}, \qquad n = 1, 2, \ldots.$$

The norming constants and eigenvalues are called spectral data of the operator $L(0,x,0)$.

Thus, we have two "model" operators on half axis with pure discrete spectra, for which we know eigenvalues, eigenfunctions and norming constants. Now we want to construct new operators (with in advance given spectra) on half axis, starting from these "model" operators.

2.2.3 On changing spectral function

The spectral function of an operator $L(0,x,0)$ is defined as [GL66, LS70]

$$\rho(\lambda) = \begin{cases} \sum_{0<\lambda_n\le\lambda} a_n^{-1}, & \lambda > 0, \\ -\sum_{\lambda<\lambda_n\le 0} a_n^{-1}, & \lambda < 0, \end{cases}$$

and $\rho(0) = 0$, i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda = \lambda_n$ equals a_n^{-1}.

In what follows $\delta(x)$ is Dirac δ-function (see, e.g. [Sch61]), δ_{ij} is Kronecker symbol and $v_{ij}(x) = \int_0^x V_i^*(s) V_j(s) ds$, where by the sign * we denote a transponation of vector functions, i.e. $\psi^*(x,\lambda) = (\psi_1(x,\lambda)\ \psi_2(x,\lambda))$, (note that $v_{ij}(x)$ is a scalar function).

In this paragraph we will answer the question, what will happen with the potential $\Omega_0(x)$ if we change spectral data, i.e., if we add or subtract eigenvalues and change the values of norming constants. It was proved (see [Har86]), that if $\rho(\lambda)$ is a spectral function of some self-adjoint operator $L(p,q,\alpha)$, then a function $\tilde{\rho}(\lambda)$, which differs from $\rho(\lambda)$ by only for finite number of points and is still remaining left-continuous, increasing, step function, is also spectral. It means that there exists a self-adjoint canonical Dirac operator $\tilde{L} = L(\tilde{p}, \tilde{q}, \alpha)$, for which $\tilde{\rho}(\lambda)$ is spectral function.

2.2.3.1 Adding and subtracting eigenvalues

At first, we want to construct a new operator $\tilde{L} = L(\tilde{p}, \tilde{q}, 0)$, which has the same spectra as $L(0, x, 0)$ except one eigenvalue. For instance, if we extract eigenvalue $\lambda_0(0, x, 0) = 0$ we will get the following

Theorem 2.2.1. *Let $\rho(\lambda)$ is a spectral function of the operator $L(0, x, 0)$. Then the function $\tilde{\rho}(\lambda)$, defined by relation*

$$\tilde{\rho}(\lambda) = \begin{cases} \rho(\lambda), & \lambda \leq \lambda_0, \\ \rho(\lambda) - a_0^{-1}, & \lambda > \lambda_0, \end{cases}$$

where $a_0 = \sqrt{\pi}/2$, i.e.

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) - \frac{1}{a_0}\delta(\lambda - \lambda_0)d\lambda \tag{2.2.18}$$

is also spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ generated by the differential expression $\tilde{l} = B\frac{d}{dx} + \tilde{\Omega}(x)$ and the boundary condition (2.2.14), for which $\tilde{\rho}(\lambda)$ is spectral function. Wherein, the potential function $\tilde{\Omega}(x)$ is represented by the following formula

$$\tilde{\Omega}(x) = \begin{pmatrix} 0 & x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2} ds} \\ x - \dfrac{e^{-x^2}}{a_0 - \int_0^x e^{-s^2} ds} & 0 \end{pmatrix} \tag{2.2.19}$$

and for the eigenfunctions the following formulae hold

$$\tilde{V}_n(x) = \begin{pmatrix} V_{n,1}(x) \\ \\ V_{n,2}(x) + \dfrac{e^{-\frac{x^2}{2}} \int_0^x e^{-\frac{s^2}{2}} V_{n,2}(s) ds}{a_0 - \int_0^x e^{-s^2} ds} \end{pmatrix}, \qquad n \in \mathbb{Z}\backslash\{0\}. \tag{2.2.20}$$

Proof. At first we denote $\tilde{\psi}(x, \lambda) = \psi(x, \lambda, 0, \tilde{\Omega})$ and $\psi(x, \lambda) = \psi(x, \lambda, 0, \Omega_0)$. It is known (see [GL66, Mar77, LS70, Har08, AHM05]), that there exists transformation operator $\mathbb{I} + \mathbb{G}$:

$$\tilde{\psi}(x,\lambda) = (\mathbb{I} + \mathbb{G})\psi(x,\lambda) = \psi(x,\lambda) + \int_0^x G(x,s)\psi(s,\lambda)ds, \tag{2.2.21}$$

which transforms the solution $\psi(x,\lambda)$ of the Cauchy problem $S(0,x,\lambda,0)$ to the solutions $\tilde{\psi}(x,\lambda)$ of the Cauchy problem $S(\tilde{p},\tilde{q},\lambda,0)$. It is also known (see, e.g. [GL66, LS70]), that the kernel $G(x,y)$ satisfies to the Gel'fand-Levitan integral equation:

$$G(x,y) + F(x,y) + \int_0^x G(x,s)F(s,y)ds = 0, \quad 0 \le y \le x < \infty, \tag{2.2.22}$$

where matrix function $F(x,y)$ is defined by the formula

$$F(x,y) = \int_{-\infty}^{\infty} \psi(x,\lambda)\psi^*(y,\lambda)d[\tilde{\rho}(\lambda) - \rho(\lambda)]. \tag{2.2.23}$$

It is also known that the potentials $\tilde{\Omega}(x)$ and $\Omega_0(x)$ are connected by the relation

$$\tilde{\Omega}(x) = \Omega_0(x) + G(x,x)B - BG(x,x). \tag{2.2.24}$$

From the (2.2.8)-(2.2.11) and definition (2.2.16) it follows, that $V_0^*(x) = (0 \quad e^{-\frac{x^2}{2}})$. Putting the relation (2.2.18) into (2.2.23), and using (2.2.17), for the kernel $F(x,y) = F_0(x,y)$ we obtain:

$$F_0(x,y) = -a_0^{-1}\psi(x,\lambda_0)\psi^*(y,\lambda_0) = -a_0^{-1}V_0(x)V_0^*(y) =$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & -a_0^{-1}e^{-\frac{(x^2+y^2)}{2}} \end{pmatrix}. \tag{2.2.25}$$

After some calculations from the equation (2.2.22) and formula (2.2.25) for $G_0(x,y)$ we obtain

$$G_0(x,y) = \frac{1}{a_0 - \int_0^x e^{-s^2}ds}V_0(x)V_0^*(y) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{e^{-\frac{(x^2+y^2)}{2}}}{a_0 - \int_0^x e^{-s^2}ds} \end{pmatrix}.$$

Now taking into account (2.2.17), putting $G_0(x,y)$ into the equations (2.2.21) and (2.2.24) we can easily obtain (2.2.19) and (2.2.20). Theorem 2.2.1 is proved. □

Now we want to subtract any finite number of eigenvalues. For this reason we denote by Z_n the arbitrary set of finite n number of integers, in increasing order, $Z_n = \{z_1, z_2, \dots, z_n\} \subset \mathbb{Z}$ (e.g., if $Z_4 = \{z_1, z_2, z_3, z_4\} = \{-127, 0, 32, 1259\}$, for $\sum_{i=1}^{4} s_{z_i} = s_{-127} + s_0 + s_{32} + s_{1259}$).

Theorem 2.2.2. *Let $\rho(\lambda)$ is the spectral function of the operator L. Then the function $\tilde{\rho}(\lambda)$, defined by relation*

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) - \sum_{k=1}^{n} a_{z_k}^{-1}\delta(\lambda - \lambda_{z_k})d\lambda$$

is also spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ generated on half axis by the differential expression $\tilde{l} = B\frac{d}{dx} + \tilde{\Omega}(x)$ and the boundary condition (2.2.14), *for which $\tilde{\rho}(\lambda)$ is spectral function. Wherein, the potential function $\tilde{\Omega}(x)$ is*

$$\tilde{\Omega}(x) = \begin{pmatrix} p(x,n) & q(x,n) \\ q(x,n) & -p(x,n) \end{pmatrix},$$

where $p(x,n)$ and $q(x,n)$ are defined by the following formulae:

$$p(x,n) = -\frac{1}{\det S(x,n)} \sum_{k=1}^{n}\sum_{p=1}^{2} V_{z_k,(3-p)}(x) \det S_p^{(k)}(x,n),$$

$$q(x,n) = x + \frac{1}{\det S(x,n)} \sum_{k=1}^{n}\sum_{p=1}^{2} (-1)^{p-1} V_{z_k,p}(x) \det S_p^{(k)}(x,n),$$

where $S(x,n)$ is $n \times n$ square matrix $S(x,n) = \{\delta_{z_i z_j} - a_{z_j}^{-1} v_{z_i z_j}(x)\}_{i,j=1}^{n}$ and $S_p^{(k)}(x,n)$ are matrices, which are obtained from the matrix $S(x,n)$, when we replace k-th column of $S(x,n)$ by $H_p(x,n) = \{a_{z_i}^{-1} V_{z_i,p}(x)\}_{i=1}^{n}$ column, $p = 1,2$. And for the eigenfunctions $\tilde{V}_m(x)$ ($m \in \mathbb{Z}\backslash Z_n$) we obtain the representations

$$\tilde{V}_m(x) = \begin{pmatrix} V_{m,1}(x) + \frac{1}{\det S(x,n)} \sum_{k=1}^{n} v_{z_k m}(x) \det S_1^{(k)}(x,n) \\ \\ V_{m,2}(x) + \frac{1}{\det S(x,n)} \sum_{k=1}^{n} v_{z_k m}(x) \det S_2^{(k)}(x,n) \end{pmatrix}.$$

Proof. In this case the kernel $F(x,y)$ can be written in the following form:

$$F(x,y) = F_n(x,y) = \sum_{k=1}^{n} -a_{z_k}^{-1} V_{z_k}(x) V_{z_k}^*(y), \tag{2.2.26}$$

and consequently, the integral equation (2.2.22) becomes to an integral equation with degenerate kernel, i.e. it becomes to a system of linear equations and we will look for the solution in the following form:

$$G_n(x,y) = \sum_{k=1}^{n} g_{z_k}(x) V_{z_k}^*(y), \tag{2.2.27}$$

where $g_{z_k}(x) = \begin{pmatrix} g_{z_k,1}(x) \\ g_{z_k,2}(x) \end{pmatrix}$ is unknown vector-function. Putting the expressions (2.2.26) and (2.2.27) into the integral equation (2.2.22) we will obtain a system of algebraic equations for determining the vector-functions $g_{z_k}(x)$:

$$g_{z_k}(x) - \sum_{i=1}^{n} a_{z_k}^{-1} v_{z_i z_k}(x) g_{z_i}(x) = a_{z_k}^{-1} V_{z_k}(x), \quad k = 1, 2, \ldots, n. \tag{2.2.28}$$

It would be better if we consider the equations (2.2.28) for the vectors $g_{z_k}(x)$ by coordinates $g_{z_k,1}(x)$ and $g_{z_k,2}(x)$ to be systems of scalar linear equations:

$$g_{z_k,p}(x) - \sum_{i=1}^{n} a_{z_k}^{-1} v_{z_i z_k}(x) g_{z_i,p}(x) = a_{z_k}^{-1} V_{z_k,p}(x), \quad k = 1, 2, \ldots, n, \quad p = 1, 2.$$

The latter systems might be written in matrix form

$$S(x,n) g_p(x,n) = H_p(x,n), \qquad p = 1, 2,$$

where the column vectors $g_p(x,n) = \{g_{z_k,p}(x,n)\}_{k=1}^{n}$, $p = 1, 2$. The solution of this system can be found in the form (Cramer's rule):

$$g_{z_k,p}(x,d) = \frac{\det S_p^{(k)}(x,n)}{\det S(x,n)}, \quad k = 1, 2, \ldots, n, \quad p = 1, 2.$$

Thus we have obtained for $g_{z_k}(x)$ the following representation:

$$g_{z_k}(x,n) = \frac{1}{\det S(x,n)} \begin{pmatrix} \det S_1^{(k)}(x,n) \\ \det S_2^{(k)}(x,n) \end{pmatrix}.$$

Using these $g_{z_k}(x,n)$, from (2.2.27) we find the function $G_n(x,y)$. Now taking into account (2.2.17), putting $G_n(x,y)$ into the equations (2.2.24) and (2.2.21) we obtain the representations for $p(x,n)$, $q(x,n)$ and $\tilde{V}_m(x)$, $m \in \mathbb{Z}\backslash\{Z_n\}$.

Theorem 2.2.2 is proved. □

Now we want to add any finite number of new real eigenvalues $\mu_k \neq \lambda_m$, $m \in \mathbb{Z}$, to the spectra, with positive norming constants c_k, $k = 1, 2, \ldots, n$.

Theorem 2.2.3. *Let $\rho(\lambda)$ is the spectral function of the operator L, then the function $\tilde{\rho}(\lambda)$, defined by relation*

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) + \sum_{k=1}^{n} c_k^{-1} \delta(\lambda - \mu_k) d\lambda$$

also is spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ generated on half axis by the differential expression $\tilde{l} = B\dfrac{d}{dx} + \tilde{\Omega}(x)$ and the boundary condition (2.2.14)*, for which $\tilde{\rho}(\lambda)$ is spectral function. Wherein, the potential function $\tilde{\Omega}(x)$ is*

$$\tilde{\Omega}(x) = \begin{pmatrix} p(x,n) & q(x,n) \\ q(x,n) & -p(x,n) \end{pmatrix},$$

where $p(x,n)$ and $q(x,n)$ are defined by the following formulae:

$$p(x,n) = -\frac{1}{\det S(x,n)} \sum_{k=1}^{n} \sum_{p=1}^{2} W_{k,(3-p)}(x) \det S_p^{(k)}(x,n),$$

$$q(x,n) = x + \frac{1}{\det S(x,n)} \sum_{k=1}^{n} \sum_{p=1}^{2} (-1)^{p-1} W_{k,p}(x) \det S_p^{(k)}(x,n),$$

and where $W_k(x) := \psi(x, \mu_k, 0, \Omega_0)$, $k = 1, 2, \ldots, n$, and $S(x,n)$ is $n \times n$ square matrix $S(x,n) = \{\delta_{ij} + c_j^{-1} w_{ij}(x)\}_{i,j=1}^{n}$ ($w_{ij}(x) = \int_0^x W_i^*(s) W_j(s) ds$), and $S_p^{(k)}(x,n)$ are matrices, which are obtained from the matrix $S(x,n)$, when we replace k-th column of $S(x,n)$ by

$H_p(x,n) = \{-c_i^{-1} W_{i,p}(x)\}_{i=1}^n$ *column,* $p = 1, 2$. *For the eigenfunctions* $\tilde{V}_m(x)$ *(for* $m \in \mathbb{Z}$ *we obtain the representations*

$$\tilde{V}_m(x) = \begin{pmatrix} V_{m,1}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{k=1}^{n} \int_0^x W_k^*(s) V_m(s) ds \det S_1^{(k)}(x,n) \\ \\ V_{m,2}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{k=1}^{n} \int_0^x W_k^*(s) V_m(s) ds \det S_2^{(k)}(x,n) \end{pmatrix},$$

and for the eigenfunctions $\tilde{W}_k(x)$ *(for* $k = 1, 2, \ldots, n$*) we obtain the representations*

$$\tilde{W}_k(x) = \begin{pmatrix} W_{k,1}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{l=1}^{n} w_{lk}(x) \det S_1^{(l)}(x,n) \\ \\ W_{k,2}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{l=1}^{n} w_{lk}(x) \det S_2^{(l)}(x,n) \end{pmatrix}.$$

The proof is similar to the proof of Theorem 2.2.2.

2.2.3.2 Scaling norming constants

The following theorem says that one can change the values of the finite number of norming constants a_n by any positive number $b_n \neq a_n$.

Theorem 2.2.4. *Let* $\rho(\lambda)$ *is the spectral function of the operator* L. *Then the function* $\tilde{\rho}(\lambda)$, *defined by relation*

$$d\tilde{\rho}(\lambda) = d\rho(\lambda) + \sum_{k=1}^{n} (b_{z_k}^{-1} - a_{z_k}^{-1}) \delta(\lambda - \lambda_{z_k}) d\lambda$$

also is spectral. Moreover, there exists unique self-adjoint canonical Dirac operator $\tilde{L}$ *generated on half axis by the differential expression* $\tilde{l} = B\dfrac{d}{dx} + \tilde{\Omega}(x)$ *and the boundary condition* (2.2.14), *for which* $\tilde{\rho}(\lambda)$ *is spectral function. Wherein, the potential function* $\tilde{\Omega}(x)$ *is*

$$\tilde{\Omega}(x) = \begin{pmatrix} p(x,n) & q(x,n) \\ q(x,n) & -p(x,n) \end{pmatrix},$$

where $p(x,n)$ *and* $q(x,n)$ *are defined by the following formulae:*

$$p(x,n) = -\frac{1}{\det S(x,n)} \sum_{k=1}^{n} \sum_{p=1}^{2} V_{z_k,(3-p)}(x) \det S_p^{(k)}(x,n),$$

$$q(x,n) = x + \frac{1}{\det S(x,n)} \sum_{k=1}^{n} \sum_{p=1}^{2} (-1)^{p-1} V_{z_k,p}(x) \det S_p^{(k)}(x,n),$$

where $S(x,n)$ is $n \times n$ square matrix $S(x,n) = \{\delta_{z_i z_j} + (b_{z_i}^{-1} - a_{z_i}^{-1}) v_{z_i z_j}(x)\}_{i,j=1}^{n}$ and $S_p^{(k)}(x,n)$ are matrices, which are obtained from the matrix $S(x,n)$, when we replace k-th column of $S(x,n)$ by $H_p(x,n) = \{-(b_{z_j}^{-1} - a_{z_j}^{-1}) V_{z_i,p}(x)\}_{i=1}^{n}$ column, $p = 1,2$. And for the eigenfunctions $\tilde{V}_m(x)$ $(m \in \mathbb{Z})$ we obtain the representations

$$\tilde{V}_m(x) = \begin{pmatrix} V_{m,1}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{k=1}^{n} v_{z_k m}(x) \det S_1^{(k)}(x,n) \\ \\ V_{m,2}(x) + \dfrac{1}{\det S(x,n)} \displaystyle\sum_{k=1}^{n} v_{z_k m}(x) \det S_2^{(k)}(x,n) \end{pmatrix}.$$

The proof is similar to the proof of Theorem 2.2.2.

Thus, we have proved, that one can perturb the linear potential of canonical Dirac operator by adding, subtracting finite number of the eigenvalues and/or changing finite number of norming constants with having changed potential function in explicit form.

Remark 2.2.1. *We take the operator $L(0,x,0)$ as a "model" operator for perturbing spectral function. Analogues theorems can be proven for the second model operator $L(0,x,\pi/2)$.*

2.3 Gradient of eigenvalues and its applications

In this section we have a goal to describe the dependence of λ_n on quantities p, q and α, β more precisely. We introduce a concept of eigenvalues' gradient, by the following formula (compare with [IT83])

$$grad\lambda_n = \left(\frac{\partial\lambda_n}{\partial\alpha}, \frac{\partial\lambda_n}{\partial\beta}, \frac{\partial\lambda_n}{\partial p(x)}, \frac{\partial\lambda_n}{\partial q(x)}\right).$$

Definition 2.3.1. *Let g is defined on (a, b), where $-\infty \leq a < b \leq \infty$. The derivative of a function f with respect to a function g is called a function $\frac{\partial f}{\partial g(x)}$, which satisfies the equation*

$$\frac{d}{d\epsilon} f(g + \epsilon v)\Big|_{\epsilon=0} = \int_a^b \frac{\partial f}{\partial g(x)} v(x) dx,$$

for all $v \in L^2_{\mathbb{R}}(a, b)$.

We want to express the components of the eigenvalues' gradient by normalized eigenfunctions of $L(p, q, \alpha, \beta)$ problem.

Theorem 2.3.1. *Let λ_n and $h_n(x)$ are eigenvalues and normalized eigenfunctions of the problem $L(p, q, \alpha, \beta)$ correspondingly. Then the following relations hold*

$$\frac{\partial\lambda_n(\alpha, \beta, p, q)}{\partial\alpha} = -|h_n(0)|^2,$$
$$\frac{\partial\lambda_n(\alpha, \beta, p, q)}{\partial\beta} = |h_n(\pi)|^2,$$
$$\frac{\partial\lambda_n(\alpha, \beta, p, q)}{\partial p(x)} = |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2,$$
$$\frac{\partial\lambda_n(\alpha, \beta, p, q)}{\partial q(x)} = 2h_{n_1}(x) \cdot h_{n_2}(x).$$

Proof. Let h_n is an eigenfunction of problem $L(p, q, \alpha, \beta)$, and $\tilde{h}_n$ is an eigenfunction of problem $L(p, q, \alpha + \Delta\alpha, \beta)$. Then

$$\ell h_n \equiv Bh_n'(x) + \Omega(x)h_n(x) \equiv \lambda_n(\alpha)h_n(x), \tag{2.3.1}$$

$$h_{n_1}(0)\cos\alpha + h_{n_2}(0)\sin\alpha = 0,$$

$$h_{n_1}(\pi)\cos\beta + h_{n_2}(\pi)\sin\beta = 0.$$

$$\ell\tilde{h}_n \equiv B\tilde{h}_n'(x) + \Omega(x)h_n(x) \equiv \lambda_n(\alpha+\Delta\alpha)\tilde{h}_n(x), \tag{2.3.2}$$

$$\tilde{h}_{n_1}(0)\cos(\alpha+\Delta\alpha) + \tilde{h}_{n_2}(0)\sin(\alpha+\Delta\alpha) = 0,$$

$$\tilde{h}_{n_1}(\pi)\cos\beta + \tilde{h}_{n_2}(\pi)\sin\beta = 0.$$

Multiplying (2.3.1) by $\tilde{h}_n(x)$ scalarly from the right, and (2.3.2) by $h_n(x)$ from the left, and taking into account the self-adjointness of $\Omega(x)$ $\Big((h_n, \Omega\tilde{h}_n) = (\Omega h_n, \tilde{h}_n)\Big)$, we obtain

$$\Big(Bh_n', \tilde{h}_n\Big) + \Big(\Omega h_n, \tilde{h}_n\Big) = \lambda_n(\alpha)\Big(h_n, \tilde{h}_n\Big),$$
$$\Big(h_n, B\tilde{h}_n'\Big) + \Big(\Omega h_n, \tilde{h}_n\Big) = \lambda_n(\alpha+\Delta\alpha)\Big(h_n, \tilde{h}_n\Big).$$

Subtracting from the second equation the first equation, we get

$$\int_0^\pi \Big\langle \begin{pmatrix} h_{n_1} \\ h_{n_2} \end{pmatrix}, \begin{pmatrix} \tilde{h}_{n_2}' \\ -\tilde{h}_{n_1}' \end{pmatrix} \Big\rangle dx - \int_0^\pi \Big\langle \begin{pmatrix} h_{n_2}' \\ -h_{n_1}' \end{pmatrix}, \begin{pmatrix} \tilde{h}_{n_1} \\ \tilde{h}_{n_2} \end{pmatrix} \Big\rangle dx =$$

$$= [\lambda_n(\alpha+\Delta\alpha) - \lambda_n(\alpha)]\Big(h_n, \tilde{h}_n\Big). \tag{2.3.3}$$

Taking into account, that in case of real potentials the components of the solutions can be taken real, thus the left side of the latter equation can be written as

$$\int_0^\pi \Big[h_{n_1}(x)\tilde{h}_{n_2}'(x) - h_{n_2}(x)\tilde{h}_{n_1}'(x) - h_{n_2}'(x)\tilde{h}_{n_1}(x) + h_{n_1}'(x)\tilde{h}_{n_2}(x)\Big]dx =$$
$$= \int_0^\pi \frac{d}{dx}\Big[h_{n_1}(x)\tilde{h}_{n_2}(x) - h_{n_2}(x)\tilde{h}_{n_1}(x)\Big]dx =$$
$$= h_{n_1}(\pi)\tilde{h}_{n_2}(\pi) - h_{n_2}(\pi)\tilde{h}_{n_1}(\pi) - h_{n_1}(0)\tilde{h}_{n_2}(0) + h_{n_2}(0)\tilde{h}_{n_1}(0).$$

Since

$$h_n(x) = \frac{\varphi_n(x,\alpha)}{\sqrt{a_n(\alpha)}}, \qquad \tilde{h}_n(x) = \frac{\varphi_n(x,\alpha+\Delta\alpha)}{\sqrt{a_n(\alpha+\Delta\alpha)}},$$

then

$$h_n(0) = \frac{1}{\sqrt{a_n(\alpha)}}\begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}, \qquad \tilde{h}_n(0) = \frac{1}{\sqrt{a_n(\alpha+\Delta\alpha)}}\begin{pmatrix} \sin(\alpha+\Delta\alpha) \\ -\cos(\alpha+\Delta\alpha) \end{pmatrix}.$$

Thus the equation (2.3.3) can be rewritten as follows

$$-\frac{1}{\sqrt{a_n(\alpha)a_n(\alpha+\Delta\alpha)}}\sin\Delta\alpha = [\lambda_n(\alpha+\Delta\alpha) - \lambda_n(\alpha)]\Big(h_n, \tilde{h}_n\Big).$$

From the latter, when $\Delta\alpha \to 0$, we obtain

$$\frac{\partial \lambda_n}{\partial \alpha} = -\frac{1}{a_n} = -|h_n(0)|^2.$$

Similarly we obtain

$$\frac{\partial \lambda_n}{\partial \beta} = \frac{1}{b_n} = |h_n(\pi)|^2.$$

To obtain the equality $\dfrac{\partial \lambda_n}{\partial p(x)} = |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2$, we write (2.3.1) in the form

$$Bh_n'(x) + (\sigma_2 p(x) + \sigma_3 q(x))\, h_n(x) \equiv \lambda_n(p) h_n(x) \tag{2.3.4}$$

and for (2.3.2) in the form

$$B\tilde{h}_n'(x) + (\sigma_2 [p(x) + \epsilon v(x)] + \sigma_3 q(x))\, \tilde{h}_n(x) \equiv \lambda_n(p+\epsilon v)\tilde{h}_n(x), \tag{2.3.5}$$

where $\tilde{h}_n$ is normalized eigenfunction of the $L(p+\epsilon v, q, \alpha, \beta)$ problem. Multiply (2.3.4) by $\tilde{h}_n(x)$ scalarly from the right, and (2.3.5) by $h_n(x)$ from the left. Taking into account, that h_n and $\tilde{h}_n$ satisfy to the same boundary conditions, subtract equality (2.3.4) from (2.3.5), we obtain

$$\left(h_n, \sigma_2 [p(x) + \epsilon v(x)]\, \tilde{h}_n\right) - \left(\sigma_2 p(x) h_n, \tilde{h}_n\right) = [\lambda_n(p+\epsilon v) - \lambda_n(p)] \left(h_n, \tilde{h}_n\right).$$

From the latter it follows

$$\frac{\lambda_n(p+\epsilon v) - \lambda_n(p)}{\epsilon} \left(h_n, \tilde{h}_n\right) = \int_0^\pi \left(h_{n_1}(x)\tilde{h}_{n_1}(x) - h_{n_2}(x)\tilde{h}_{n_2}(x)\right) v(x) dx.$$

Tending $\epsilon \to 0$, using the fact, that $\tilde{h}_n \to h_n$, when $\epsilon \to 0$ and the definition 2.3.1, we obtain $\dfrac{\partial \lambda_n}{\partial p(x)} = |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2$.

Similarly we can obtain the equality $\dfrac{\partial \lambda_n}{\partial q(x)} = 2h_{n_1}(x) \cdot h_{n_2}(x)$.

Theorem 2.3.1 is proved. □

Let us consider also canonical Dirac system on half axis. In this case the gradient is defined as

$$grad \lambda_n = \left(\frac{\partial \lambda_n}{\partial \alpha}, \frac{\partial \lambda_n}{\partial p(x)}, \frac{\partial \lambda_n}{\partial q(x)}\right),$$

and in Definition 2.3.1 we take $a = 0,\ b = \infty$.

Theorem 2.3.2. *Let* λ_n *and* $h_n(x)$ *are eigenvalues and normalized eigenfunctions of the problem* $L(p,q,\alpha)$ *correspondingly. Then the following relations hold*

$$\frac{\partial \lambda_n(\alpha,p,q)}{\partial \alpha} = -|h_n(0)|^2,$$
$$\frac{\partial \lambda_n(\alpha,p,q)}{\partial p(x)} = |h_{n_1}(x)|^2 - |h_{n_2}(x)|^2,$$
$$\frac{\partial \lambda_n(\alpha,p,q)}{\partial q(x)} = 2h_{n_1}(x)\cdot h_{n_2}(x).$$

Proof. In case of real potentials the components of the solutions can be taken real. Since the eigenfunctions h_n and $\tilde{h}_n$ are from $L^2(0,\infty)$, we can infer that the scalar products $\langle h_n, \tilde{h}_n\rangle$ are from $L^1(0,\infty)$ and, hence, are tending to 0 on some $\{x_k;\ x_k\to\infty,\ k\to\infty,\}$ sequence. Taking into account the above mentioned we prove the third formula (the first two formulae can be proved in the similar way as in Theorem 2.3.1).

Write the equation (2.3.1) in the following form

$$Bh_n'(x) + (\sigma_2 p(x) + \sigma_3 q(x))\, h_n(x) \equiv \lambda_n(p)h_n(x) \tag{2.3.6}$$

and (2.3.2) in the form

$$B\tilde{h}_n'(x) + (\sigma_2 p(x) + \sigma_3\,[q(x)+\epsilon v(x)])\,\tilde{h}_n(x) \equiv \lambda_n(p+\epsilon v)\tilde{h}_n(x), \tag{2.3.7}$$

where $\tilde{h}_n$ is normalized eigenfunction of the $L(p, q+\epsilon v, \alpha, \beta)$ problem. Multiplying (2.3.6) scalarly by $\tilde{h}_n(x)$ from the right, and (2.3.7) by $h_n(x)$ from the left. Taking into account, that h_n and $\tilde{h}_n$ satisfy to the same boundary condition, subtracting equality (2.3.6) from (2.3.7), we obtain

$$\left(h_n, \sigma_3\,[q(x)+\epsilon v(x)]\,\tilde{h}_n\right) - \left(\sigma_3 q(x) h_n, \tilde{h}_n\right) = [\lambda_n(q+\epsilon v) - \lambda_n(q)]\left(h_n, \tilde{h}_n\right).$$

From the latter equation we have

$$\begin{aligned}\int_0^\infty \left(h_{n_1}(x)\bar{\tilde{h}}_{n_2}(x) + h_{n_2}(x)\bar{\tilde{h}}_{n_1}(x)\right)\epsilon v(x)dx = \\ = [\lambda_n(q+\epsilon v) - \lambda_n(q)]\left(h_n, \tilde{h}_n\right).\end{aligned} \tag{2.3.8}$$

And from the equation (2.3.8) it follows

$$\frac{\lambda_n(q+\epsilon v)-\lambda_n(q)}{\epsilon}\left(h_n, \tilde{h}_n\right) = \int_0^\infty \left(h_{n_1}\bar{\tilde{h}}_{n_2} + h_{n_2}\bar{\tilde{h}}_{n_1}\right)v(x)dx.$$

Tending $\epsilon \to 0$, using the fact, that $\tilde{h}_n \to h_n$, when $\epsilon \to 0$ and the definition 2.3.1, we obtain $\dfrac{\partial \lambda_n}{\partial q(x)} = 2h_{n_1}(x)h_{n_2}(x)$.

Theorem 2.3.2 is proved. □

2.3.1 On isospectral operators on a finite interval

It is well-known, that the inverse problem of reconstruction of operator $L(p,q,\alpha,\beta)$ by spectral function (in our case by eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ and norming constants $\{a_n\}_{n\in\mathbb{Z}}$) can not be solved uniquely, if we permit parameters α and β to be arbitrary (see [GL66]). But if we fix one of them, then the inverse problem can be solved uniquely (see [GL66, AHM05, Har94, Wat99]). Therefore, usually is considered the problem $L(p,q,\alpha,0)$ (see [GD75, AHM05, Har94, AH17]).

It is also well-known, that for regular Dirac operators (the operators on finite interval with summable coefficients), we can not add or diminish the eigenvalues (because of obligatory asymptotics (2.1.4)), staying in the class of summable coefficients, but we can change the norming constants and describe the isospectral Dirac operators (see [Har94, AH17]).

The applications of eigenvalues' gradient of describing operators, which isospectral with fixed operator $L(p,q,\alpha,0)$ is given in section 2.3.1. On the other hand, if we consider Dirac operator on half axis $(0,\infty)$ (which has pure discrete spectra), we can add or diminish arbitrary finite number of eigenvalues or change norming constants, since in this case there are no obligatory asymptotics (see, e.g. [AH16]). The applications of eigenvalues' gradient in this case is given in section 2.3.2.

Consider the boundary-value problem $L(p,q,\alpha,0)$ on $[0,\pi]$. From the eigenvalues' asymptotics (2.1.4) it follows:

$$\lambda_n(\Omega,\alpha,0) = n - \frac{\alpha}{\pi} + r_n, \quad r_n = o(1), \quad n \to \pm\infty. \tag{2.3.9}$$

It is known (see [GD75, HA06]) that in the case of $\Omega \in L^2_{\mathbb{R}}[0,\pi]$ the norming constants have an asymptotic form:

$$a_n(\Omega) = \pi + c_n, \qquad \sum_{n=-\infty}^{\infty} c_n^2 < \infty.$$

Our main goal is to give the description of the set $M^2(\Omega)$ in terms of eigenvalues' gradients.

If we denote

$$\frac{\partial \lambda_n}{\partial \Omega(x)} := \begin{pmatrix} \dfrac{\partial \lambda_n}{\partial p(x)} & \dfrac{\partial \lambda_n}{\partial q(x)} \\ \dfrac{\partial \lambda_n}{\partial q(x)} & -\dfrac{\partial \lambda_n}{\partial p(x)} \end{pmatrix} = \begin{pmatrix} h_{n_1}^2(x) - h_{n_2}^2(x) & 2h_{n_1}(x)h_{n_2}(x) \\ 2h_{n_1}(x)h_{n_2}(x) & -(h_{n_1}^2(x) - h_{n_2}^2(x)) \end{pmatrix},$$

we will have

$$B\frac{\partial \lambda_n}{\partial \Omega(x)} = \begin{pmatrix} 2h_{n_1}(x)h_{n_2}(x) & h_{n_2}^2(x) - h_{n_1}^2(x) \\ h_{n_2}^2(x) - h_{n_1}^2(x) & -2h_{n_1}(x)h_{n_2}(x) \end{pmatrix}. \tag{2.3.10}$$

And it is easy to see, that the term $[Bh_{\tilde{m}}(x, \Omega(\cdot, T_{m-1}))h_{\tilde{m}}^*(\cdot) - h_{\tilde{m}}(\cdot)h_{\tilde{m}}^*(\cdot)B]$ of $\triangle\Omega(x, T_m)$ is equal to $B\dfrac{\partial \lambda_n}{\partial \Omega(x, T_m)}$. Therefore Theorems 2.1.2 and 2.1.3 can be rewritten as

Theorem 2.3.3. *Let $t \in \mathbb{R}$, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then*

$$\Omega(x, t) = \Omega(x) + \frac{(e^t - 1)}{\theta_m(x, t, \Omega)} B\frac{\partial \lambda_m}{\partial \Omega(\cdot, T_m))},$$

where $\theta_m(x, t, \Omega) = 1 + (e^t - 1)\int_0^x |h_m(s, \Omega)|^2 ds$. So, for arbitrary $t \in \mathbb{R}$, $\lambda_n(\Omega, t) = \lambda_n(\Omega)$ for all $n \in \mathbb{Z}$, $a_n(\Omega, t) = a_n(\Omega)$ for all $n \in \mathbb{Z}\backslash\{m\}$ and $a_m(\Omega, t) = a_m(\Omega)e^{-t}$.

Theorem 2.3.4. *Let $T = \{t_n, n \in \mathbb{Z}\} \in l^2$ and $\Omega \in L^2_{\mathbb{R}}[0, \pi]$. Then*

$$\Omega(x, T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \frac{e^{t_{\tilde{m}}} - 1}{\theta_m(x, t_{\tilde{m}}, \Omega(x, T_{m-1}))} B\frac{\partial \lambda_{\tilde{m}}}{\partial \Omega(x, T_{m-1}))}.$$

2.3.2 On changing spectral data on half axis

Let us consider canonical Dirac operator $L(p, q, \alpha)$ on $[0, \infty)$, which has a pure discrete spectra. In work [Har86], Harutyunyan proved, that in this case one can add or subtract a finite number of eigenvalues, or scale the values of norming constants (i.e. a_n to change by $e^t a_n$, for arbitrary $t \in \mathbb{R}$). In that work explicit formulae for potential functions of changed operator are given.

According to the paper [Har86], when we want to add a new eigenvalue μ, the formula for potential function $\Omega_1(x)$ will be:

$$\Omega_1(x) \equiv \Omega(x) + \frac{1}{1 + \int_0^x |h(t,\mu)|^2 dt}\{Bh(x,\mu)h^*(x,\mu) - h(x,\mu)h^*(x,\mu)B\}. \tag{2.3.11}$$

When we want to subtract an eigenvalue, e.g. λ_0, the formula for potential function $\Omega_2(x)$ will be:

$$\Omega_2(x) \equiv \Omega(x) - \frac{1}{1 - \int_0^x |h(t,\lambda_0)|^2 dt}\{Bh(x,\lambda_0)h^*(x,\lambda_0) - h(x,\lambda_0)h^*(x,\lambda_0)B\}. \tag{2.3.12}$$

When we want to scale the value of a norming constant, e.g. a_0, which corresponds to eigenvalue λ_0, the formula for potential function $\Omega_3(x)$ will be:

$$\Omega_3(x) \equiv \Omega(x) + \frac{e^{-t} - 1}{1 + (e^{-t} - 1)\int_0^x |h(t,\lambda_0)|^2 dt}\{Bh(x,\lambda_0)h^*(x,\lambda_0) - h(x,\lambda_0)h^*(x,\lambda_0)B\}. \tag{2.3.13}$$

Using formula (2.3.10) we can rewrite the formulae (2.3.11)–(2.3.13) in terms of eigenvalues' gradient:

$$\Omega_1(x) \equiv \Omega(x) + \frac{1}{1 + \int_0^x |h(t,\mu)|^2 dt} \cdot \frac{\partial \mu}{\partial \Omega(x)},$$
$$\Omega_2(x) \equiv \Omega(x) - \frac{1}{1 - \int_0^x |h(t,\lambda_0)|^2 dt} \cdot \frac{\partial \lambda_0}{\partial \Omega(x)},$$
$$\Omega_3(x) \equiv \Omega(x) + \frac{e^{-t} - 1}{1 + (e^{-t} - 1)\int_0^x |h(t,\lambda_0)|^2 dt} \cdot \frac{\partial \lambda_0}{\partial \Omega(x)}.$$

In [Har86] there is also given a formula for changing finite number of eigenvalues or norming constants. If we want to add n number of eigenvalues μ_k, to subtract m number of eigenvalues λ_k and to scale l number of norming constants a_k, then the formula for such potential $\tilde{\Omega}(x)$ depending of initial potential $\Omega(x)$ will be:

$$\tilde{\Omega}(x) \equiv \Omega(x) + \sum_{k=1}^{n+m+l} \frac{\gamma_k}{1 + \gamma_k \int_0^x |h(t,\Omega_{k-1},\nu_k)|^2 dt} \cdot$$
$$\cdot \{Bh(x,\Omega_{k-1},\nu_k)h^*(x,\Omega_{k-1},\nu_k) - h(x,\Omega_{k-1},\nu_k)h^*(x,\Omega_{k-1},\nu_k)B\}. \tag{2.3.14}$$

where

$$\gamma_k = \begin{cases} 1, & 1 \leq k \leq n, \\ -1, & n+1 \leq k \leq n+m, \\ e^{-t} - 1, & n+m+1 \leq k \leq n+m+l, \end{cases}$$

$$\nu_k = \begin{cases} \mu_k, & 1 \leq k \leq n, \\ \lambda_k, & n+1 \leq k \leq n+m+l, \end{cases}$$

and potential function $\Omega_0(x) = \Omega(x)$ and $\Omega_k(x)$, for $k = 0, 1, \ldots, n+m+l$, are given by formula:

$$\Omega_k(x) = \Omega_{k-1}(x) + \frac{\gamma_k}{1 + \gamma_k \int_0^x |h(t, \Omega_{k-1}, \nu_k)|^2 dt} \cdot$$
$$\cdot \{Bh(x, \Omega_{k-1}, \nu_k)h^*(x, \Omega_{k-1}, \nu_k) - h(x, \Omega_{k-1}, \nu_k)h^*(x, \Omega_{k-1}, \nu_k)B\}.$$

Using formula (2.3.10) we can rewrite the (2.3.14) in terms of eigenvalues' gradient:

$$\tilde{\Omega}(x) \equiv \Omega(x) + \sum_{k=1}^{n+m+l} \frac{\gamma_k}{1 + \gamma_k \int_0^x |h(t, \Omega_{k-1}, \nu_k)|^2 dt} \cdot \frac{\partial \nu_k}{\partial \Omega_{k-1}(x)}$$

Conclusion

The main results of the current thesis are:

1. The necessary and sufficient conditions for two sequences $\{\lambda_n^2\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ to be correspondingly the set of eigenvalues and the set of norming constants of a Sturm-Liouville problem with a real summable potential q and in advance fixed separated boundary conditions are found.

2. Connections of the set of norming constants and boundary parameters α and β are found.

3. A uniqueness theorem with the lowest eigenvalue $\mu_0(q, \alpha, \beta)$ for inverse Sturm-Liouville problem is proved.

4. The bounds for the lowest eigenvalue $\mu_0(q, \alpha, \beta)$ Sturm-Liouville problem are found.

5. It is shown, that there is a set of boundary conditions for which a generalization of Ambarzumyan's theorem is valid.

6. New kind of uniqueness theorems, in some sense a generalization of Marchenko's theorem, conditioned by inequalities for inverse Sturm-Liouville problem are provided and proved. And other uniqueness theorems with inequalities are proved.

7. The description of all isospectral Dirac operators (on a finite interval), in explicit form and only in terms of the normalized eigenfunctions of the initial operator, is given.

8. The eigenvalues and eigenfunctions for Dirac operators with linear potential on whole and half axes are found in explicit form.

9. Perturbations of Dirac operators with linear potential (when one add or subtract finite number of eigenvalues and scale the values of norming constants) are constructed.

10. The concept of eigenvalues' gradient is given and formulae for this gradient are obtained on a finite interval and on an half axis.

11. The concept of eigenvalues' derivative with respect to canonical matrix-potential is introduced and shown how it is used for describing the isospectral Dirac operators or when finite number of spectral data is changed.

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