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# ON SOME CLASSES OF NON-OSCILLATORY HOMOGENEOUS SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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## Abstract

The linear homogeneous system of first order differential equations with constant coefficients is considered here. The non-oscillatory is proved for those classes of homogeneous system of first order linear ordinary differential equations which coefficients are the successive members of some arithmetic or geometrical progressions.

## 1. Introduction and Auxiliary Propositions

Research of different properties of solutions of the differential equations and homogeneous linear systems of differential equations remains in sign of many mathematicians until now. Special interest cause for researchers the questions related to oscillation and non-oscillation solutions of the mentioned equations and systems (see, for example, [3]-[15]). The aim of the real work

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is to prove non-oscillation for some classes of the homogeneous systems of ordinary differential equations.

The systems to be considered in this paper are the form

$$\vec{y}' = A(t)\vec{y}, \tag{1.1}$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdots \\ y_n(t) \end{pmatrix},$$

$a_{ij}(t)(i,\ j = 1, 2, ..., n)$ are real, continuous functions on the whole numerical line. Let $A_i(t)(A'(t)_i),\ i = 1, 2, ..., n$ denote by $i$th column ($i$th row) of matrix $A(t).$ Then matrix can be written in the following type:

$$A(t) = [A_1(t),\ A_2(t),\ ...,\ A_n(t)]$$

or

$$A(t) = [A_1'(t);\ A_2'(t);\ ...,\ A_n'(t)].$$

**Definition 1.1.** A nontrivial solution $\vec{y}$ of system (1.1) is said to be *oscillatory* if each of his components has at least one zero in any neighbourhood of $+\infty$ and $-\infty.$

**Definition 1.2.** System (1.1) is said to be *oscillatory* if all its nontrivial solutions are oscillatory, otherwise it is said to be *non-oscillatory*.

**Lemma 1.1.** *Let matrix* $A$ $(n \geq 3)$ *such, that for any three numbers* $i \neq j \neq k,$ $(i,\ j,\ k = 1,\ 2,\ ...,\ n),$ *a condition*

$$A_i(t) - A_j(t) = A_j(t) - A_k(t)(A_i'(t) - A_j'(t) = A_j'(t) - A_k'(t)),$$

*is executed. Then matrix A is singular.*

**Proof.** Follows from linearly dependence of columns (rows) of matrix $A$.

**Lemma 1.2.** *Let matrix $A$ $(n \geq 2)$ such, that his elements at least of some two rows (columns) are by the successive members of geometrical progressions with equal denominators. Then matrix $A$ is singular.*

Really, we will suppose that in a matrix $A(t)$ the property indicated in a lemma is possessed by columns $A_i(t)$ and $A_j(t)$, $j \neq i$, and, consequently, it is possible to write them down in a kind

$$A_i(t) = (a_i(t) a_i(t) q(t) \dots a_i(t) q^{n-1}(t)),$$

$$A_j(t) = (a_j(t) a_j(t) q(t) \dots a_j(t) q^{n-1}(t)).$$

Then, taking $a_i(t)$ away from the *i*th column, and $a_j(t)$ from the *j*th column, we will get a determinant in which two columns *i*th and *j*th will coincide, and, consequently, his determinant and determinant of matrix $A(t)$ will appear identical equal to the zero.

**Theorem 1.1.** *If in matrix* $A(t)(n \geq 3)$,

$$A_i'(t) = [\varphi_i(t),\ \varphi_i(t) + \psi_i(t),\ \varphi_i(t) + 2\psi_i(t),\ \dots,\ \varphi_i(t) + (n-1)\psi_i(t)],$$

$$i = 1, 2, \dots, n,$$

*then the set of eigenvalues of matrix* $A(t_0)(t_0 \in [a, b])$ *contains a zero of multiplicity not less than* $n - 2$.

**Proof.** We assume that

$$A_i'(t) = [\varphi_i(t),\ \varphi_i(t) + \psi_i(t),\ \varphi_i(t) + 2\psi_i(t),\ \dots,\ \varphi_i(t) + (n-1)\psi_i(t)],$$

$$i = 1, 2, \dots, n.$$

Let us check the correctness of the given lemma foremost at $n = 3$. It is not difficult to find, that the characteristic equation of $A$ at this value will look like

$$\Phi(\lambda, t_0)$$

$$= \begin{vmatrix} \varphi_1(t_0) - \lambda & \varphi_1(t_0) + \psi_1(t_0) & \varphi_1(t_0) + 2\psi_1(t_0) \\ \varphi_2(t_0) & \varphi_2(t_0) + \psi_2(t_0) - \lambda & \varphi_2(t_0) + 2\psi_2(t_0) \\ \varphi_3(t_0) & \varphi_3(t_0) + \psi_3(t_0) & \varphi_3(t_0) + 2\psi_3(t_0) - \lambda \end{vmatrix}$$

$$= -\lambda^3 + (\varphi_1(t_0) + (\varphi_2(t_0) + \psi_2(t_0)) + (\varphi_3(t_0) + 2\psi_3(t_0)))\lambda^2$$

$$-\left( \begin{vmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \psi_1(t_0) & \psi_2(t_0) \end{vmatrix} + \begin{vmatrix} \varphi_2(t_0) & \varphi_3(t_0) \\ \psi_2(t_0) & \psi_3(t_0) \end{vmatrix} + 2 \begin{vmatrix} \varphi_1(t_0) & \varphi_3(t_0) \\ \psi_1(t_0) & \psi_3(t_0) \end{vmatrix} \right)\lambda,$$

and, hence, multiplicity of eigenvalue $\lambda = 0$ of matrix $A$ is greater or equal to one. In general case, the characteristic equation of $A$ will look like

$$\Phi(\lambda, t) = \det(A(t) - \lambda E)$$

$$= \begin{vmatrix} \varphi_1(t_0) - \lambda & \varphi_1(t_0) + \psi_1(t_0) & \varphi_1(t_0) + 2\psi_1(t_0) & \cdots & \varphi_1(t_0) + (n-1)\psi_1(t_0) \\ \varphi_2(t_0) & \varphi_2(t_0) + \psi_2(t_0) - \lambda & \varphi_2(t_0) + 2\psi_2(t_0) & \cdots & \varphi_2(t_0) + (n-1)\psi_2(t_0) \\ \varphi_3(t_0) & \varphi_3(t_0) + \psi_3(t_0) & \varphi_3(t_0) + 2\psi_3(t_0) - \lambda & \cdots & \varphi_3(t_0) + (n-1)\psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_n(t_0) & \varphi_n(t) + \psi_n(t) & \varphi_n(t_0) + 2\psi_n(t_0) & \cdots & \varphi_n(t_0) + (n-1)\psi_n(t_0) - \lambda \end{vmatrix}.$$

We first note, that $\lambda = 0$ is the root of the characteristic equation of $A$, because by Lemma 1.1 $\Phi(t_0, 0) = \det A(t_0) = 0.$ Further, it is obvious, that the coefficient of $\lambda^n$ will be equal $(-1)^n$, and coefficient of $\lambda^{n-1}$ will be equal

$$(-1)^{n-1}[\varphi_1(t_0) + (\varphi_2(t_0) + \psi_2(t_0)) + \ldots + (\varphi_н(t_0) + (n-1)\psi_n(t_0))]$$

$$= (-1)^{n-1}\sum_{k=1}^{n}(\varphi_k(t_0) + (k-1)\psi_k(t_0)).$$

We will show now, that $\lambda = 0$ is the root of characteristic equation of $A$, and it has multiplicity not less than $\lambda = 0$. For this purpose, we will transform the first determinant of $\Phi(t, \lambda)$, subtracting from every column of matrix, beginning from the second and previous columns and writing down it in place of subtrahend. According to well-known properties of determinants (see, for example, [1]), its value will not change thus and we will get as a result

$$\Phi(t_0, \lambda)$$

$$= \det(A(t_0) - \lambda E)$$

$$= \begin{vmatrix} \varphi_1(t_0) - \lambda & \psi_1(t_0) + \lambda & \psi_1(t_0) & \cdots & \psi_1(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) - \lambda & \psi_2(t_0) + \lambda & \cdots & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) - \lambda & \cdots & \psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_n(t_0) & \psi_n(t_0) & \psi_n(t_0) & \cdots & \psi_n(t_0) - \lambda \end{vmatrix}.$$

Using the formula for the definition of a derivative of the determinant, we obtain

$$\Phi'_\lambda(t_0, \lambda)$$

$$= \begin{vmatrix} -1 & 1 & 0 & \cdots & 0 \\ \varphi_2(t_0) & \psi_2(t_0) - \lambda & \psi_2(t_0) + \lambda & \cdots & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) - \lambda & \cdots & \psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_n(t_0) & \psi_n(t_0) & \psi_n(t_0) & \cdots & \psi_n(t_0) - \lambda \end{vmatrix}$$

$$+ \begin{vmatrix} \varphi_1(t_0) - \lambda & \psi_1(t_0) + \lambda & \psi_1(t_0) & \cdots & \psi_1(t_0) \\ 0 & -1 & 1 & \cdots & 0 \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) - \lambda & \cdots & \psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_n(t_0) & \psi_n(t_0) & \psi_n(t_0) & \cdots & \psi_n(t_0) - \lambda \end{vmatrix}$$

$$+\cdots+\begin{vmatrix} \varphi_1(t_0)-\lambda & \psi_1(t_0)+\lambda & \psi_1(t_0) & \dots & \psi_1(t_0) \\ \varphi_2(t_0) & \psi_2(t_0)-\lambda & \psi_2(t_0)+\lambda & \dots & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0)-\lambda & \dots & \psi_3(t_0) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}. \quad (1.2)$$

In case $n = 4,$ this formula will look like

$$\Phi'_\lambda(t, \lambda) \equiv \begin{vmatrix} -1 & 1 & 0 & 0 \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t) \end{vmatrix}$$

$$+\begin{vmatrix} \varphi_1(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ 0 & -1 & 1 & 0 \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) \end{vmatrix}$$

$$+\begin{vmatrix} \varphi_1(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ 0 & 0 & -1 & 1 \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) \end{vmatrix}$$

$$+\begin{vmatrix} \varphi_1(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

On the right, in this expression, in the second determinant we summarize the 2nd and the 3rd columns, writing down a result in place of the 2nd column, and then we will take away 2 for a determinant. In the third determinant, we will do the indicated actions with the 3rd and the 4th columns, writing down a result in place of the third column. Taking into account properties of determinants (see, for example, [1]), we will get

$$\Phi'_\lambda(t,\,0) \equiv \begin{vmatrix} -1 & 1 & 0 & 0 \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) \end{vmatrix}$$

$$+ 2\cdot \begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) \\ 0 & 0 & 1 & 0 \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) \end{vmatrix}$$

$$+ 2\cdot \begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ 0 & 0 & 0 & 1 \\ \varphi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) & \psi_4(t_0) \end{vmatrix}$$

$$+ \begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) \\ 0 & 0 & 0 & -1 \end{vmatrix}.$$

All determinants in the right resulting expression will appear equal to the zero, as each of them has two identical columns.

In general case, when $n > 4$, as a result of substitution $\lambda = 0$ in expression (1.2), we will have

$$\Phi'_\lambda(t,\,0) \equiv \begin{vmatrix} -1 & 1 & 0 & \ldots & 0 \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \ldots & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \ldots & \psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_n(t_0) & \psi_n(t_0) & \psi_n(t_0) & \ldots & \psi_n(t_0) \end{vmatrix}$$

$$+ \begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) & \ldots & \psi_1(t_0) \\ 0 & -1 & 1 & \ldots & 0 \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \ldots & \psi_3(t_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varphi_n(t_0) & \psi_n(t_0) & \psi_n(t_0) & \ldots & \psi_n(t_0) \end{vmatrix}$$

$$+\cdots+\begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \psi_1(t_0) & \cdots & \psi_1(t_0) \\ \varphi_2(t_0) & \psi_2(t_0) & \psi_2(t_0) & \cdots & \psi_2(t_0) \\ \varphi_3(t_0) & \psi_3(t_0) & \psi_3(t_0) & \cdots & \psi_3(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}.$$

These expressions imply that $\Phi'(t_0, \lambda)|_{\lambda=0} \equiv 0,$ since in all determinants on the right will have at least two identical columns, and therefore, they all will be equal to zero.

Further, we also note, that in expression (1.2) each of the determinants on the right contains on one row with permanent numbers. From here, and from the rule of differentiation of the determinant, it follows that $\Phi''_{\lambda\lambda}(t_0, \lambda)$ will contain determinants at that one of rows zero (and, consequently, they are equal to the zero), or the two rows of that consist of permanent numbers, thus in this case even in one row alongside numbers will appear 1 and –1. At a substitution $\lambda = 0,$ the determinants of one of these three types will appear on the right, namely:

1. The determinants in which at least two columns coincide.

2. The determinants, in which by summing the elements of two adjacent columns and the imposition of 2, we will get determinants with two coinciding columns.

3. The determinants, in which by summing the elements of the three adjacent columns and the imposition of 3, we will get again two coinciding columns in the determinants.

In each of these cases, the got determinants will be equal to the zero. It will be necessary from here, that $\Phi''_{\lambda\lambda}(t_0, \lambda) = 0$ if $n > 4$ and $\lambda = 0.$ Continuing reasoning by a foregoing method, we will conclude, that $\Phi^{(n-3)}(t_0, \lambda)|_{\lambda=0}$ will contain the determinants of one of these three foregoing types and, consequently, all will be equal to the zero. We have showed thus, that then multiplicity of eigenvalue $\lambda = 0$ of matrix $A(t)$ is greater or equal to $n - 2,$ what was required to show.

Note, that it is possible with simple calculations to define and coefficient at $\lambda^{n-2}$ in characteristic equation of matrix $A(t)$ or, that the same $\frac{1}{(n-2)!}\Phi^{(n-2)}(t_0, \lambda)|_{\lambda=0}$. The higher conducted reasoning show that at a calculation $\Phi^{(n-2)}(t_0, \lambda)|_{\lambda=0}$, we will run into the determinants of kind

$$\begin{vmatrix} \varphi_1(t_0) & \psi_1(t_0) & \cdots & \psi_1(t_0) & \psi_1(t_0) & \cdots & \psi_1(t_0) \\ 1 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_\kappa(t_0) & \psi_\kappa(t_0) & \cdots & \psi_k(t_0) & \psi_\kappa(t_0) & \cdots & \psi_\kappa(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -1 \end{vmatrix},$$

in which $n-2$ rows contain numbers 0, 1 and –1, and two rows look like $A'_\kappa(t_0) = (\varphi_\kappa(t_0)\psi_k(t_0)\psi_k(t_0)\ldots\psi_k(t_0)\psi_k(t_0))$. It is easy to show that the value of each of these determinants it is possible to present in a kind $(-1)^n k \cdot \begin{vmatrix} \varphi_i(t_0) & \psi_i(t_0) \\ \varphi_{i+k}(t_0) & \psi_{i+k}(t_0) \end{vmatrix}$, $k = 1, 2, ..., n-1$; $i = 1, 2, ..., n-k$. And hence, for $\Phi^{(n-2)}(t_0, \lambda)|_{\lambda=0}$ we will have

$$\Phi^{(n-2)}(t_0, \lambda)|_{\lambda=0} = (-1)^n \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k \cdot \begin{vmatrix} \varphi_i(t_0) & \psi_i(t_0) \\ \varphi_{i+k}(t_0) & \psi_{i+k}(t_0) \end{vmatrix}.$$

Generalizing the conducted reasoning, we will get, that the characteristic equation for matrix $A(t_0)$ in examined case will look like

$$\det(A(t_0) - \lambda E) = (-1)^n \lambda^{n-2}(\lambda^2 - b(t_0)\lambda + c(t_0)), \tag{1.3}$$

where

$$b(t_0) = \sum_{k=1}^{n} (\varphi_k(t_0) + (k-1)\psi_k(t_0)),$$

$$c(t_0) = \frac{1}{(n-2)!} \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k \cdot \begin{vmatrix} \varphi_i(t_0) & \psi_i(t_0) \\ \varphi_{i+k}(t_0) & \psi_{i+k}(t_0) \end{vmatrix}. \tag{1.4}$$

From (1.3) and (1.4), it follows that if

$$c(t_0) = \sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k \cdot \begin{vmatrix} \varphi_i(t_0) & \psi_i(t_0) \\ \varphi_{i+k}(t_0) & \psi_{i+k}(t_0) \end{vmatrix} \equiv 0,$$

then multiplicity of eigenvalue $\lambda = 0$ is equal, $n-1$, and if it will appear equal to the zero the $b(t_0)$, then multiplicity of $\lambda = 0$ will be equal to *n*. Note, that it is possible, in particular, if

$$\varphi_i(t_0) = \psi_i(t_0)(i = 1, 2, ..., n),$$

$$\varphi_1(t_0) + 2\varphi_2(t_0) + \cdots + n\varphi_n(t_0) = spA(t_0) = 0. \tag{1.5}$$

**Corollary 1.1.** *If considering the elements of matrix A, from the first element of the first row to the last element of the last* (*moving on rows*), *are the successive members of some arithmetic progression, then the set of eigenvalues of matrix A contains a zero of multiplicity* $n-2$ *and two real numbers of opposite signs*.

Really, we will suppose that elements of matrix are the successive members of some arithmetic progression with an initial member equal to *a* and with a difference *d*. For determination of characteristic equation, we will notice that in this case according to formula (1.4), we will have

$$b = \sum_{k=1}^{n}(a_k + (k-1)d_k)$$

$$= \sum_{k=1}^{n}(a + (k-1)nd + (k-1)d) = na + \frac{(n-1)n(n+1)}{2}d,$$

$$c = \frac{1}{(n-2)!}\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k \cdot \begin{vmatrix} a_i & d_i \\ a_{i+k} & d_{i+k} \end{vmatrix}$$

$$= \frac{1}{(n-2)!}\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k \cdot \begin{vmatrix} a + (i-1)nd & d \\ a + (i+k-1)nd & d \end{vmatrix}$$

$$= -\frac{1}{(n-2)!}\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} k^2 nd^2 = -\frac{n}{(n-2)!}d^2\sum_{k=1}^{n-1}(n-k)k^2.$$

Consequently, characteristic equation for a matrix $A$ in examined case it is possible to present in a kind

$$\det(A - \lambda E) = (-1)^n \lambda^{n-2}[\lambda^2 - (na + m)\lambda + c],$$

where $m = \frac{(n-1)n(n+1)}{2} d$, $c = -\frac{nd^2}{(n-2)!} \sum_{k=1}^{n-1} (n-k)k^2$, from where claim of investigation will follow.

**Theorem 1.2.** *If every row of matrix* $A(t)$ *can be presented in a kind*

$$A_i(t) = (a_i(t) a_i(t) q(t) \ldots a_i(t) q^{n-1}(t)), \quad i = 1, 2, \ldots, n,$$

*then set of eigenvalues of matrix* $A(t_0)(t_0 \in [a, b])$ *contains a zero multiplicity* $n-1$ *and one real number.*

**Proof.** We will consider foremost the rightness of statement at $n = 3$. We will find a direct calculation, that characteristic equation of matrix $A(t_0)$ at this value will look like

$$\Phi(t_0, \lambda) = \begin{vmatrix} a_1(t_0) - \lambda & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) \\ a_2(t_0) & a_2(t_0)q(t_0) - \lambda & a_2(t_0)q^2(t_0) \\ a_3(t_0) & a_3(t_0)q & a_3(t_0)q^2(t_0) - \lambda_0 \end{vmatrix}$$

$$= -\lambda^3 + (a_1(t_0) + a_2(t_0)q(t_0) + a_3(t_0)q^2(t_0))\lambda^2,$$

and, hence, multiplicity of eigenvalue $\lambda = 0$ of matrix $A(t_0)$ is greater or equal to two.

In general case, the characteristic equation of $A$ will look like

$$\Phi(t_0, \lambda)$$

$$= \det(A(t_0) - \lambda E)$$

$$= \begin{vmatrix} a_1(t_0)-\lambda & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) & \cdots & a_1(t_0)q^{n-1}(t_0) \\ a_2(t_0) & a_2(t_0)q(t_0)-\lambda & a_2(t_0)q^2(t_0) & \cdots & a_2(t_0)q^{n-1}(t_0) \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0)-\lambda & \cdots & a_3(t_0)q^{n-1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n(t_0) & a_n(t_0)q(t_0) & a_n(t_0)q^2(t_0) & \cdots & a_n(t_0)q^{n-1}(t_0)-\lambda \end{vmatrix}.$$

Concordantly Lemma 1.2, $\lambda = 0$ is the root of characteristic equation of $A(t_0)$. Further, it is obvious, that by a coefficient in $\lambda^n$ a number $(-1)^n$ will appear and coefficient in $\lambda^{n-1}$ will be equal $\lambda^{n-1}$

$$(-1)^{n-1}[a_1(t_0) + (a_2(t_0)q(t_0)) + \ldots + (a_н(t_0)q^{n-1}(t_0))]$$

$$= (-1)^{n-1}\sum_{k=1}^{n} a_k(t_0)q^{k-1}(t_0).$$

We will show now, that $\lambda = 0$ is the root of characteristic equation of $A(t_0)$ multiplicity not less than $n-1$. Using the definition of derivative of determinant, we obtain

$$\varphi'(\lambda)$$

$$= \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ a_2(t_0) & a_2(t_0)q(t_0) - \lambda & a_2(t_0)q^2(t_0) & \cdots & a_2(t_0)q^{n-1}(t_0) \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0) - \lambda & \cdots & a_3(t_0)q^{n-1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n(t_0) & a_n(t_0)q(t_0) & a_n(t_0)q^2(t_0) & \cdots & a_n(t_0)q^{n-1}(t_0) - \lambda \end{vmatrix}$$

$$+ \begin{vmatrix} a_1(t_0) & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) & \ldots & a_1(t_0)q^{n-1}(t_0) \\ 0 & -1 & 0 & \ldots & 0 \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0) - \lambda & \ldots & a_3(t_0)q^{n-1}(t_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n(t_0) & a_n(t_0)q(t_0) & a_n(t_0)q^2(t_0) & \ldots & a_n(t_0)q^{n-1}(t_0) - \lambda \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} a_1(t_0) & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) & \ldots & a_1(t_0)q^{n-1}(t_0) \\ a_2(t_0) & a_2(t_0)q(t_0) - \lambda & a_2q^2 & \ldots & a_2(t_0)q^{n-1}(t_0) \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3q^2 - \lambda & \ldots & a_3(t_0)q^{n-1}(t_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -1 \end{vmatrix}. \quad (1.6)$$

Putting $\lambda = 0$ in expression (1.6), we will have

$$\Phi'(t_0, \lambda)$$

$$= \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ a_2(t_0) & a_2(t_0)q(t_0) & a_2(t_0)q^2(t_0) & \cdots & a_2(t_0)q^{n-1}(t_0) \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0) & \cdots & a_3(t_0)q^{n-1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n(t_0) & a_n(t_0)q(t_0) & a_n(t_0)q^2(t_0) & \cdots & a_n(t_0)q^{n-1}(t_0) \end{vmatrix}$$

$$+ \begin{vmatrix} a_1(t_0) & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) & \ldots & a_1(t_0)q^{n-1}(t_0) \\ 0 & -1 & 0 & \ldots & 0 \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0) & \ldots & a_3(t_0)q^{n-1}(t_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n(t_0) & a_n(t_0)q(t_0) & a_n(t_0)q^2(t_0) & \ldots & a_n(t_0)q^{n-1}(t_0) \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} a_1(t_0) & a_1(t_0)q(t_0) & a_1(t_0)q^2(t_0) & \cdots & a_1(t_0)q^{n-1}(t_0) \\ a_2(t_0) & a_2(t_0)q(t_0) & a_2(t_0)q^2(t_0) & \cdots & a_2(t_0)q^{n-1}(t_0) \\ a_3(t_0) & a_3(t_0)q(t_0) & a_3(t_0)q^2(t_0) & \cdots & a_3(t_0)q^{n-1}(t_0) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix},$$

from where it will be necessary, that $\Phi'(t_0, 0) = 0$, because in all determinants will appear on the right at two such columns (we will suppose the *i*th and the *j*th), that at taking $q^{i-1}(t_0)$ away from the *i*th column and $q^{j-1}(t_0)$ from the *j*th, we will get two identical columns, and that these determinants will appear equal to the zero.

Further, we will notice also, that in expression (1.7) each of determinants on the right contain one row with permanent numbers. From here, and from the rule of differentiation of determinant, it will be necessary that $\Phi''(t_0, \lambda)$ will contain determinants in which the rows are zero (and, consequently, they are equal to the zero), or two rows of that consist of $n - 2$ zeros and one –1.

At a substitution $\lambda = 0$, will appear on the right determinants in that at least two columns which contain two zeros. Then, repeating foregoing procedure with these columns, we will get determinants with two consilient columns, and, consequently, they also will be equal to the zero. Continuing reasoning by a foregoing method, we will come to the conclusion, that $\Phi^{(n-2)}(t_0, \lambda)$ also will be equal to the zero. And consequently, the multiplicity of eigenvalue $\lambda = 0$ of matrix $A$ will be greater or equal $n - 1$, what was required to show.

By virtue of the conducted reasoning the characteristic equation of matrix $A$ will look like

$$\det(A - \lambda E) = (-1)^n \lambda^{n-1}\left(\lambda - \sum_{k=1}^{n} a_\kappa q^{k-1}\right). \tag{1.7}$$

**Corollary 1.2.** *If considering the elements of matrix A, from the first element of the first row to the last element of the last row* (*moving on rows*), *are the successive members of some geometrical progression*, *then the set of eigenvalues of matrix A contains the zero of multiplicity* $n - 1$.

Really, in examined case $a_1(t_0) = a$, and, consequently, $a_k(t_0) = aq^{kn-1}$. Then, according to (1.7), the characteristic equation of matrix $A$ will look

$$\det(A - \lambda E) = (-1)^n \lambda^{n-1}\left(\lambda - a\sum_{k=1}^{n} q^{k-1}\right) = (-1)^n \lambda^{n-1}\left(\lambda - a\,\frac{q^n - 1}{q - 1}\right).$$

## 2. The Main Results

**Theorem 2.1.** *If elements of every row of matrix A are successive members of some arithmetic progressions*, *then the system* (1.1) *is non-oscillatory*.

**Proof.** It is known (see, for example, [5]) that the common solution of the system (1.1) can be presented in a kind

$$\vec{\varphi}(t) = \sum_{k=1}^{m} \vec{g}_k(t) e^{\lambda_k t}, \tag{2.1}$$

where $\lambda_\kappa$ $(k = 1, 2, ..., m)$-different from each other eigenvalues of matrix $A$, and coordinates of vector-function $\vec{g}_k(t)$ are the polynomials of degree not higher than $r_k - 1$, where $r_k$-multiplicity of eigenvalue $\lambda_\kappa$. According to claim of Corollary 1.1, the eigenvalues of matrix $A$ will be zero and some numbers $\lambda_1$ and $\lambda_2$. Then the common solution of the system (1.1), concordantly (2.1), it is possible to write down in a kind

$$\vec{\varphi}(t) = c_1 \vec{p}_1 e^{\lambda_1 t} + c_2 \vec{p}_2 e^{\lambda_2 t} + \vec{g}(t),$$

where $\vec{g}(t)$ -is vector-function, which components are polynomials of degree not higher than $n - 3$, $c_1$, $c_2$ -are arbitrary constants, $\vec{p}_1$, $\vec{p}_2$ are eigenvectors of matrix $A$, corresponding eigenvaules $\lambda_1$ and $\lambda_2$. It is obvious, that at $t \to +\infty$ the module of components will aspire to infinity and, consequently, the system cannot have an oscillatory solution.

**Theorem 2.2.** *If the elements of matrix A satisfy to the terms of Lemma* 1.1, *then one component of particular solution of the system* (1) *will be polynomial of degree not higher than* $n - 3$.

**Proof.** It is known (see, for example, [2]), that particular solutions of the system (1.1), corresponding eigenvalue $\lambda$ of matrix $A$, it is possible to present in a kind

$$\vec{\varphi}(t) = \vec{g}(t) e^{\lambda t}, \tag{2.2}$$

in which coordinates of vector-function $\vec{g}(t)$ are polynomials of degree not higher than $r_k - 1$, where $r_k$ is multiplicity of eigenvalue $\lambda$. According to claim of Lemma 1.1, zero is the eigenvalue of matrix $A$ and he has multiplicity not less than $n - 2$. Hence, particular solutions of system (1.1), corresponding eigenvalue $\lambda = 0$, by (2.2), will look like

$$\vec{\varphi}(t) = \vec{g}(t),$$

where coordinates of vector-function $\vec{g}(t)$ are polynomials of degree not higher than, $n-3$, what was required to show.

**Corollary 2.1.** *If the elements of matrix A satisfy to the terms of Lemma* 1.2, *and also terms* (1.5), *then components of every solution of the system* (1.1) *will be polynomial of degree not higher than* $n-1$.

**Theorem 2.3.** *If elements of every row of matrix A are the successive members of some geometrical progressions with equal denominators*, *then the system* (1.1) *is non-oscillatory*.

**Proof.** Taking advantage of formula (2.1), and also, taking into account claim of Lemma 1.2, we will get, that the common decision of the system (1.1) can be presented in a kind

$$\vec{\varphi}(t) = c\vec{p}e^{\lambda t} + \vec{g}(t),$$

where $\lambda$ is eigenvalue of matrix $A$ different from a zero, $c$ is arbitrary permanent, $\vec{p}$ is eigenvector of matrix $A$ corresponding eigenvalue $\lambda$, and coordinates of vector-function $\vec{g}(t)$ are polynomials of degree not greater than $n-2$. It is obvious, that when $t \to +\infty$ the module of components will aspire to infinity, and, consequently, the system (1.1) is non-oscillatory.

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