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M a t h e m a t i c s

ON SOLVABILITY OF SOME BOUNDARY PROBLEM

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We constract the exact solution of the Dirichlet problem in the Sobolev space for two-dimensional elliptic equation considered on the half-plane.

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Problem Setup. Let $S'(\mathbb{R}^n)$ be the space of tempered distributions (slowly growing). It is well known that the Fourier transform

$$\hat{u}(\lambda) = (Fu)(\lambda) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} u(x)\, e^{i(\lambda_1 x_1 + \cdots + \lambda_n x_n)} dx,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $x = (x_1, \ldots, x_n)$, $dx = dx_1, \ldots, dx_n$, is a linear continuous isomorphism on $L_2(\mathbb{R}^n)$ and can be continuously extended to a topological isomorphism of the space $S'(\mathbb{R}^n)$ on itself (see, for exasmple, [1]).

Consider the Sobolev spaces

$$H^s(\mathbb{R}^n) := \left\{ u \in S'(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \, (|\xi|^2 + 1)^s \, d\xi < \infty \right\},$$

$$H^s(\mathbb{R}^n_+) := \Big\{ u \in S'(\mathbb{R}^n) : \operatorname{supp} u \subset \bar{\mathbb{R}}^n_+, \ u = v \ \text{ on } \ \mathbb{R}^n_+ \ \text{ for some } \ v \in H^s(\mathbb{R}^n_+) \Big\},$$

where $s \in \mathbb{R}$, $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $|\xi|^2 = \sum\limits_{i=1}^{n} \xi_i^2$.

The space $H^s(\mathbb{R}^n)$ with the norm $\|u\|_{H^s(\mathbb{R}^n)} := \left\|(1+|x|^2)^{s/2}\, \hat{u}\right\|_{H^s(\mathbb{R}^n)}$, is a Hilbert space. The space $H^s(\mathbb{R}^n_+)$ with the norm $\|u\|_{H^s(\mathbb{R}^n_+)} := \inf \|v\|_{H^s(\mathbb{R}^n)}$, where the inf is taken over all continuations of u belonging to $H^s(\mathbb{R}^n)$, is also a Hilbert space (see [1] or [2]).

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For a natural number s the Hilbert space $H^s(\Omega)$ of functions, where $\Omega \subset \mathbb{R}^n$, coincides with the set of functions $\mathcal{D}^\alpha u$, where $u \in L_2(\Omega)$, satisfies generalized derivatives $\mathcal{D}^\alpha u \in L_2(\Omega)$ for all $1 \leqslant |\alpha| \leqslant s$. Here $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Fourier image of the space $H^s(\mathbb{R}^n)$ will be denoted by $\hat{H}^s(\mathbb{R}^n)$,

$$\|\hat{u}\|_{\hat{H}^s(\mathbb{R}^n)} := \|u\|_{H^s(\mathbb{R}^n)}.$$

Let $\mathcal{D}(\bar{\mathbb{R}}^n_+)$ and $\mathcal{D}(\mathbb{R}^n)$ be the sets of infinitely differentiable functions with a compact support defined respectively on $\bar{\mathbb{R}}^n_+$ and $\mathbb{R}^n$.

It is known (see [1]) that $\mathcal{D}(\bar{\mathbb{R}}^n_+) \subset H^1(\mathbb{R}^n_+)$ and the mapping $u(x) \mapsto u(x', 0)$ $(x = (x', x_n),\ x' \in \mathbb{R}^{n-1})$ acting from $\mathcal{D}(\bar{\mathbb{R}}^n_+)$ to $\mathcal{D}(\mathbb{R}^{n-1})$ can be continuously extended to the operator $T_+ : H^1(\mathbb{R}^n_+) \to H^{1/2}(\mathbb{R}^{n-1})$. For $u \in H^1(\mathbb{R}^n_+)$ the functions $T_+ u$ commonly called trace of the function u.

Observe that (see [2]) any function $u(x', x_n) \in H^1(\mathbb{R}^n)$ is continuous with respect to $x \in \mathbb{R}^n$ and takes values in $H^{1/2}(\mathbb{R}^{n-1})$, i.e.

$$\left\| u(x', x_n + y_n) - u(x', x_n) \right\|_{H^{1/2}(\mathbb{R}^{n-1})} \longrightarrow 0$$

as $y_n \to 0$.

The papers [3–5] (see also [6]) investigated the diffraction problem of Sommerfeld with boundary conditions of the first and second kind in the Sobolev space.

Our investigations are based on an explicit solution of the "weak" Dirichlet problem of the equation

$$\Delta u + k^2 u = 0, \tag{1}$$

considered in the space $H^1(\mathbb{R}^2_+)$, where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, $\quad k \in \mathbb{C}$ and $\operatorname{Im} k > 0$. The latter claim to find the solution of Eq. (1) in $H^1(\mathbb{R}^2_+)$ with a given trace.

In this paper we solve an analogous problem for the equation

$$(\Delta + k^2 + \beta(y))u = 0, \tag{2}$$

where $\operatorname{Im} k > 0$ and $\beta(y) = -2/\operatorname{ch}^2 y$.

Dirichlet Problem. Denote by $t(\lambda) = (\lambda^2 - k^2)^{1/2}$ the analytical branch of the square root on the plane with an incision along the rays $\{\pm k \pm i\omega;\ \omega \geqslant 0\}$ tending to $+\infty$ as $\lambda \to \pm\infty$.

The next theorem allows us to restore the solution of Eq. (2) in the class $H^1(\mathbb{R}^2_+)$ with a given trace.

Theorem . *If a function g belongs to $H^{1/2}(\mathbb{R})$, then Eq.* (2) *has a unique solution in the space $H^1(\mathbb{R}^2_+)$, which satisfies to the boundary condition*

$$u_0^+(x) := u(x, +0) = g(x). \tag{3}$$

This solution is given by the formula

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{g}(\lambda) \left(1 + \frac{\operatorname{th} y}{t(\lambda)}\right) e^{-t(\lambda) y} d\lambda. \tag{4}$$

Proof. We apply the Fourier transform to Eq. (2) with respect to the variable x. Let

$$v(\lambda,y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} u(x,y)\,dx. \tag{5}$$

The function v satisfies to the following ordinary differential equation

$$\frac{d^2v}{dy^2}(\lambda,y) - Q(\lambda,y)v(\lambda,y) = 0, \quad \lambda \in \mathbb{R},\ y > 0, \tag{6}$$

where $Q(\lambda,y) = t^2(\lambda) + \beta(y) = \lambda^2 - k^2 - 2/\mathrm{ch}^2 y$. It is easy to check that the functions

$$v_1(\lambda,y) = (t(\lambda) - \mathrm{th}\,y)e^{t(\lambda)y},$$
$$v_2(\lambda,y) = (t(\lambda) + \mathrm{th}\,y)e^{-t(\lambda)y}$$

with respect to the variable y form a fundamental system of solutions for Eq. (6).

Let function $u \in H^1(\mathbb{R}^2_+)$ be a solution of Eq. (2).

It is known (see [1]) that the function u belongs to $H^1(\mathbb{R}^2_+)$ if and only if the following conditions hold:

$$\int_0^{\infty} \|u(x,y)\|^2_{H^1(\mathbb{R}_x)}\,dy < \infty, \tag{7}$$

$$\int_0^{\infty} \left\| \frac{\partial u}{\partial y}(x,y) \right\|^2_{L_2(\mathbb{R}_x)} dy < \infty. \tag{8}$$

The extension of (7) has the form

$$\int_0^{\infty}\int_{-\infty}^{\infty} (1+|\lambda|^2) \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} u(x,y)\,dx \right|^2 d\lambda\,dy =$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty} (1+|\lambda|^2)\,|v(\lambda,y)|^2\,d\lambda\,dy < \infty,$$

where the function v is defined by (5). Using Fubin's theorem, we get

$$\int_{-\infty}^{\infty} (1+|\lambda|^2) \int_0^{\infty} |v(\lambda,y)|^2\,dy\,d\lambda < \infty.$$

From here, in particular, it follows that for almost every $\lambda \in \mathbb{R}$ the function $v(\lambda,y)$ belongs to $L_2(\mathbb{R}_y)$. Thus, we can state that for the solution $u \in H^1(\mathbb{R}^2_+)$ of Eq. (2) the function v (which is the solution of Eq. (6)) defined by formula (5) has the following form

$$v(\lambda,y) = c(\lambda)v_2(\lambda,y) = c(\lambda)(t(\lambda) + \mathrm{th}\,y)\,e^{-t(\lambda)y}. \tag{9}$$

The equality

$$\lim_{y\to 0} \|u(x,y) - g(x)\|_{H^{1/2}(\mathbb{R})} = 0$$

is equivalent to the equality

$$\lim_{y\to 0}\|v(\lambda,y)-\hat{g}(\lambda)\|_{\hat{H}^{1/2}(\mathbb{R})}=0$$

and, thus, it follows from (9) that

$$c(\lambda)=\frac{\hat{g}(\lambda)}{t(\lambda)}.$$

Thus we conclude, that if the solution of the problem (2), (3) exists, then it can be uniquely represented by formula (4).

It remains to verify that for $g\in H^{1/2}(\mathbb{R})$ the function u, which is defined by (4), belongs to $H^1(\mathbb{R}^2_+)$.

Define functions w_i, $i=1,2,3$:

$$w_1(x,y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\hat{g}(\lambda)\,e^{-t(\lambda)y}d\lambda,$$

$$w_2(x,y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\,\frac{\hat{g}(\lambda)}{t(\lambda)}\,e^{-t(\lambda)y}d\lambda,$$

$$w_3(x,y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\hat{g}(\lambda)\,t(\lambda)\,e^{-t(\lambda)y}d\lambda.$$

The function w_1 is a solution of (1), (3) in the class $H^1(\mathbb{R}^2_+)$ (see [3], [5])

$$\int_0^{\infty}\|w_1(x,y)\|^2_{H^1(\mathbb{R}_x)}\,dy<\infty. \tag{10}$$

Note that

$$|t(\lambda)|\geqslant \mathrm{Re}\,t(\lambda)=$$

$$=\sqrt{\lambda^2-(\mathrm{Re}\,k)^2+(\mathrm{Im}\,k)^2+\sqrt{(\lambda^2-(\mathrm{Re}\,k)^2+(\mathrm{Im}\,k)^2)^2+4\mathrm{Re}\,k\cdot\mathrm{Im}\,k}}\geqslant\sqrt{2}\,\mathrm{Im}\,k.$$

Using the Parseval equality, we get

$$\int_0^{\infty}\|w_2(x,y)\|^2_{H^1(\mathbb{R}_x)}\,dy=\int_0^{\infty}\int_{-\infty}^{\infty}\left|\frac{\hat{g}(\lambda)}{t(\lambda)}\,e^{-t(\lambda)y}\right|^2(1+\lambda^2)\,d\lambda\,dy\leqslant$$

$$\leqslant M_1\int_0^{\infty}e^{-(2\sqrt{2}\,\mathrm{Im}\,k)y}\int_{-\infty}^{\infty}|\hat{g}(\lambda)|^2\,d\lambda\,dy\leqslant\frac{M_1}{2\sqrt{2}\,\mathrm{Im}\,k}\,\|g\|^2_{L^2(\mathbb{R})}\ , \tag{11}$$

where $M_1=\sup\limits_{\lambda\in\mathbb{R}}\dfrac{1+\lambda^2}{|t(\lambda)|^2}$. Since the function u defined by (4) satisfies the equality $u(x,y)=w_1(x,y)+\mathrm{th}\,y\,w_2(x,y)$, it satisfies condition (7). It is easy to verify that the generalized derivative

$$\frac{\partial u}{\partial y}(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\,\hat{g}(\lambda)\left(\frac{1}{t(\lambda)\mathrm{ch}^2y}-t(\lambda)-\mathrm{th}\,y\right)e^{-t(\lambda)y}\,d\lambda=$$

$$=\frac{1}{\mathrm{ch}^2y}\,w_2(x,y)-\mathrm{th}\,y\,w_1(x,y)-w_3(x,y). \tag{12}$$

Let

$$\tilde{w}_k(\lambda,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda x}w_k(x,y)\,dx,\qquad k=1,2,3.$$

Then, using the Parseval equality, we get

$$\int_0^\infty \|w_1(x,y)\|^2_{L_2(\mathbb{R}_x)}\,dy=\int_0^\infty \|\tilde{w}_1(\lambda,y)\|^2_{L_2(\mathbb{R}_\lambda)}\,dy=\int_0^\infty\int_{-\infty}^\infty \left|\hat{g}(\lambda)\,t^{-t(\lambda)y}\right|^2 dx\,dy=$$

$$=\int_0^\infty e^{-(2\sqrt{2}\,\mathrm{Im}\,k)y}\int_{-\infty}^\infty |g(x)|^2\,dx\,dy\leqslant\frac{1}{2\sqrt{2}\,\mathrm{Im}\,k}\,\|g\|_{L_2(\mathbb{R})}\,;$$

$$\int_0^\infty \|w_2(x,y)\|^2_{L_2(\mathbb{R}_x)}\,dy=\int_0^\infty \|\tilde{w}_2(\lambda,y)\|^2_{L_2(\mathbb{R}_\lambda)}\,d\lambda=$$

$$=\frac{1}{2(\mathrm{Im}\,k)^2}\int_0^\infty e^{-(2\sqrt{2}\,\mathrm{Im}\,k)y}\int_{-\infty}^\infty |\hat{g}(\lambda)|^2\,d\lambda\,dy\leqslant\frac{1}{4\sqrt{2}\,(\mathrm{Im}\,k)^3}\,\|g\|^2_{L_2(\mathbb{R})}\,;$$

$$\int_0^\infty \|w_3(x,y)\|^2_{L_2(\mathbb{R}_x)}\,dy=\int_0^\infty \|\tilde{w}_3(\lambda,y)\|^2_{L_2(\mathbb{R}_\lambda)}\,d\lambda=\int_0^\infty\int_{-\infty}^\infty \left|\hat{g}(\lambda)\,t(\lambda)\,e^{-t(\lambda)y}\right|^2 d\lambda\,dy\leqslant$$

$$\leqslant M_2\int_0^\infty \left|\hat{g}(\lambda)\,e^{-t(\lambda)y}\right|^2(1+\lambda^2)\,d\lambda\,dy\leqslant M_2\int_0^\infty \|w_1(x,y)\|^2_{H^1(\mathbb{R}_x)}\,dy<\infty,$$

where

$$M_2=\sup_{\lambda\in\mathbb{R}}\frac{|t(\lambda)|^2}{1+\lambda^2}\,.$$

It follows from these inequalities and formula (11) that the conditions (8) hold. The Proof is complete. □

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ՄԻ ԵԶՐԱՅԻՆ ԽՆԴՐԻ ԼՈՒԾԵԼԻՈՒԹՅԱՆ ՄԱՍԻՆ

Աշխատանքում կառուցված է կիսահարթության մեջ դիտարկվող մի երկչափանի էլիպտիկական հավասարման Դիրիխլեի խնդրի բացահայտ լուծումը Սոբոլևի տարածությունում:

А. Г. КАМАЛЯН, М. И. КАРАХАНЯН

О РАЗРЕШИМОСТИ ОДНОЙ ГРАНИЧНОЙ ЗАДАЧИ

В работе построено явное решение задачи Дирихле в пространстве Соболева одного двумерного эллиптического уравнения, рассматриваемого на полуплоскости.