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Mathematics

ON A GENERALIZED FORMULA OF TAYLOR–MACLAURIN TYPE ON THE GENERALIZED COMPLETELY MONOTONE FUNCTIONS

B. A. SAHAKYAN *

Chair of General Mathematics YSU, Armenia

In the paper Taylor–Maclaurin type formulas for some classes of functions are obtained. The main result of this study introduces an idea of the generalized classes of $\langle\rho_j\rangle$ completely monotone function. Under the various conditions $\left(\sum\limits_{j=1}^{\infty}\frac{1}{\rho_j}<+\infty,\ \sum\limits_{j=1}^{\infty}\frac{1}{\rho_j}=+\infty,\right)$ the terms of their representation are obtained and some related theorems are proved.

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Introduction. In the present paper are considered a system of functions

$$\left\{\frac{(a-x)^{\lambda_n}}{\Gamma(1+\lambda_n)}\right\}_0^{\infty} \tag{1°}$$

and a system of operators

$$\left\{A_{a,n}f\right\}_0^{\infty},\left\{A_{a,n}^{*}f\right\}_0^{\infty}, A_{a,n}f(x)=\prod_{j=0}^{n-1}D_a^{1/\rho_j}f(x), A_{a,n}^{*}f(x)=D_a^{-\alpha_n}A_{a,n}f(x), n\geq 1, \tag{2°}$$

$A_{a,0}f\equiv f,\ x\in[0,a],\ \lambda_n=\sum\limits_{j=1}^{n}\dfrac{1}{\rho_j},\ \rho_j\geq 1,\ \rho_0=1,\ \lambda_0=0,\ 1-\alpha_j=\dfrac{1}{\rho_j},\ j=1,2,\ldots;$

$D_a^{-\alpha_j}f(x)=\dfrac{1}{\Gamma(\alpha_j)}\displaystyle\int_x^a(t-x)^{\alpha_j-1}f(t)dt,\ D_a^{1/\rho_j}f(x)=\dfrac{d}{dx}D_a^{-\alpha_j}f(x),\ j=0,1,\ldots$

Note that in the papers [1–8] by author and prof. Dzhrbashyan were obtained various generalized formulas of Taylor–Maclaurin type, using operators of Riemann–Liouville type of fraction order and functions of Mittag–Leffler type. In these papers it was introduced the concept of absolutely monotone functions $\langle\rho\rangle$, $\langle\rho_j\rangle$, $\langle\rho,\ \lambda_j\rangle$, $\langle\rho_j,\ W_j\rangle$ and the problems of their representation were studied.

In the present paper we introduce the concept of generalized completely monotone functions $\langle\rho_j\rangle$ and study the problems of their representation. Note that using some other operators prof. Badalyan has introduced the concept of generalized regular monotone functions [9].

* E-mail: maneat@rambler.ru

We introduce general classes $\langle \rho_j \rangle$ of completely monotone functions and prove some representation theorems under the conditions:

a) $\sum\limits_{j=1}^{\infty} 1/\rho_j = +\infty, f(x) = \sum\limits_{k=0}^{\infty} (-1)^k A^*_{a,k} f(a) \dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}$, $x \in [0,a]$;

b) $\sum\limits_{j=1}^{\infty} 1/\rho_j < +\infty$, $f(x) = \sum\limits_{k=0}^{\infty} (-1)^k A^*_{a,k} f(a) \dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)} - \dfrac{1}{\Gamma(1+\lambda_\infty)} \int_x^a (t-x)^{\lambda_\infty} d\mu(t)$, $x \in (0,a]$, $\lambda_\infty = \sum\limits_{j=1}^{\infty} \dfrac{1}{\rho_j}$.

Preliminary Information and Lemmas. Let $f(x)$ be an arbitrary function from $L(0,l)$ $(0 < l < +\infty)$. The function

$$ {}_0D^{-\alpha} f(x) \equiv D^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad x \in (0,l), \tag{1}$$

is called the Riemann–Liouville integral of the function $f(x)$ of order α $(0 < \alpha < +\infty)$ with a lower limit at the point $x = 0$, and the function

$$ D_l^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^l (t-x)^{\alpha-1} f(t)dt, \quad x \in (0,l), \tag{2}$$

is called the Riemann–Liouville integral of function $f(x)$ of order α with an upper limit at the point $x = l$.

At each Lebesgue point of the function $f(x)$ and consequently almost everywhere on $(0,l)$, we have $\lim\limits_{\alpha \to +0} D^{-\alpha} f(x) = f(x)$, so we define $D^0 f(x) = f(x)$, $x \in (0,l)$. Let $\rho \geq 1$, $1/\rho = 1-\alpha$ $(0 \leq \alpha < 1)$. Then the function

$$ {}_0D^{1/\rho} f(x) \equiv D^{1/\rho} f(x) \equiv \frac{d}{dx} D^{-\alpha} f(x) \tag{3}$$

is called Riemann–Liouville derivative of order $1/\rho$ for $f(x)$ with lower limit at $x = 0$.

$D_l^{1/\rho} f(x) \equiv \dfrac{dD_l^{-\alpha} f(x)}{dx}$ is called the derivative of the function $f(x)$ with upper limit at $x = l$ (see details in [10], Chap. 9).

The function of Mittag–Leffler type

$$ E_\rho(z,\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n\rho^{-1})}, \quad \rho > 0, \tag{4}$$

is an entire function of order ρ and type 1 for any value of the parameter μ [10].

For any $\mu > 0$, $\alpha > 0$ the following formula holds:

$$ \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} E_\rho(\lambda t^{1/\rho},\ \mu) t^{\mu-1} dt = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho},\ \mu+\alpha), \tag{5}$$

where λ is a complex parameter and the integration is taken over a curve connecting the points 0 with z ([10], Eq. (1.16)).

Lemma A. [1]. Let $\varphi(x) \in L(0,l)$ $(0 < l < +\infty)$ and let λ be an arbitrary parameter. In the class of functions $y(x) \in L(0,l)$, $D^{1/\rho} y(x) \in L(0,l)$ the problem of Cauchy type

$$ D^{1/\rho} y(x) + \lambda y(x) = \varphi(x),\ x \in (0,l); \quad D^{-\alpha} y(x)\big|_{x=0} = 0, \tag{6}$$

has a unique solution $Y(x,\lambda)$, which can be expressed in the form

$$ Y(x,\lambda) = \int_0^x e_\rho(x-t,\lambda)\varphi(t)dt, x \in (0,l), e_\rho(x,\lambda) = E_\rho(-\lambda x^{1/\rho}, 1/\rho) x^{1/\rho - 1}. \tag{7}$$

From the Lemma A it particularly follows that in case $\varphi(x)\equiv 0$ we have

$$D^{1/\rho}y(x)+\lambda y(x)=0,\quad D^{-\alpha}y(x)\big|_{x=0}=0,$$

and the problem of Caushy type has a unique solution $y(x)\equiv 0$. The following lemma is a version of Lemma A.

Lemma 1. Let $\varphi(x)\in L(0,a),\quad 0<a<+\infty$, and λ be an arbitrary parameter. Then in the class of functions satisfying the conditions $y(x)\in L(0,a)$, and $D_a^{1/\rho}y(x)\in L(0,a)$ the problem of Cauchy type

$$D_a^{1/\rho}y(x)-\lambda y(x)=\varphi(x),\quad x\in(0,a),\ D_a^{-\alpha}y(x)\big|_{x=a}=0, \tag{8}$$

has a unique solution $Y(x,\lambda)$, which can be expressed in the form

$$Y(x,\lambda)=-\int_x^a e_\rho(t-x,\lambda)\varphi(t)dt. \tag{9}$$

From (9) for $\lambda=0$ we get $y(x)=-\dfrac{1}{\Gamma(1/\rho)}\displaystyle\int_x^a (t-x)^{1/\rho-1}\varphi(t)dt.$

Lemma 2. Let $\varphi(x)\in L(0,a),\ \rho\geq 1,\ 1-\alpha=1/\rho$. Then in the class of functions $y(x)\in L(0,a),\ y'(x)\in L(0,a),\ D_a^{1/\rho}y'(x)\in L(0,a)$ Cauchy type problem

$$D_a^{1/\rho}y'(x)=\varphi(x),\ x\in(0,a),\ y(x)\big|_{x=a}=0,\ \ y'(x)\big|_{x=a}=0,\ \ D_a^{-\alpha}y'(x)\big|_{x=a}=0, \tag{10}$$

has a unique solution

$$y(x)=\frac{1}{\Gamma(1+1/\rho)}\int_x^a (t-x)^{1/\rho}\varphi(t)dt. \tag{11}$$

2. Formula of Taylor–Maclaurin Type. Let sequences $\{\rho_k\}_0^\infty$, $\rho_0=1$; $\{\alpha_k\}_0^\infty$, $\alpha_0=0$; $\{\lambda_k\}_0^\infty$, $\lambda_0=0$ satisfy the conditions

$$\rho_k\geq 1,\quad \alpha_k=1-1/\rho_k,\quad \lambda_k=\sum_{j=1}^k 1/\rho_j,\quad k=1,2,\ldots \tag{12}$$

Consider the sequence of operators on an admissible class of functions $f(x)$. $\{A_{a,n}f(x)\}_0^\infty,\ \ \{A^*_{a,n}f(x)\}_0^\infty$ defined by

$$\begin{gathered}A_{a,n}f(x)=\prod_{j=0}^{n-1}D_a^{1/\rho_j}f(x)\ (A_{a,0}f\equiv f,\ \ A_{a,1}f\equiv f'(x)),\ \ n\geq 0,\\ A^*_{a,n}f(x)=D_a^{-\alpha_n}A_{a,n}f(x),\ \ A^*_{a,0}f\equiv f,\end{gathered} \tag{13}$$

where $D_a^{1/\rho_j}f(x)\equiv\dfrac{d}{dx}D_a^{-\alpha_j}f(x), D_a^{-\alpha_j}f(x)=\dfrac{1}{\Gamma(\alpha_j)}\displaystyle\int_x^a(t-x)^{\alpha_j-1}f(t)dt,\ j=0,1,...$

Consider the sequence of functions

$$\left\{\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}\right\}_0^\infty,\qquad x\in[0,a]. \tag{14}$$

Note that operators like (13) and a system like (14) were introduced in [2].

Lemma 3. Let $\varphi(x)\in L(0,a)$. Then in class of functions $A_{a,k}y(x)\in L(0,a)$ $\big(A_{a,0}y(x)\equiv y(x), A_{a,1}y(x)=y'(x)\big)$ the problem of Cauchy type

$$A_{a,n+1}y'(x)=\varphi(x),\ y(x)\big|_{x=a}=0,\ \ D_a^{-\alpha_k}A_{a,k}y'(x)\big|_{x=a}=0\ \ (k=0,1,\ldots,n) \tag{15}$$

has a unique solution $Y(x)$, which can be expressed in the form

$$Y(x)=\frac{(-1)^{n+1}}{\Gamma(1+\lambda_n)}\int_x^a(t-x)^{\lambda_n}\varphi(t)dt,\quad \lambda_n=\sum_{k=1}^n\frac{1}{\rho_k}. \tag{16}$$

Notice, that for $n=1,2$ Lemma 3 is true according to Lemma 2.

Proof. We carry out an induction argument to prove of the Lemma. Assuming that Lemma 3 is true, we show that the problem of Cauchy type

$$\begin{gathered} A_{a,n+2}y'(x)=\varphi(x),\ \ x\in(0,a),y(x)\big|_{x=a}=0, \\ D_a^{-\alpha_k}A_{a,k}y(x)\big|_{x=a}=0,\ \ k=0,1,2,\dots,n,n+1, \end{gathered} \tag{17}$$

has a unique solution

$$y(x)=\frac{(-1)^{n+2}}{\Gamma(1+\lambda_{n+1})}\int_x^a(t-x)^{\lambda_{n+1}}\varphi(t)dt. \tag{18}$$

Note that $A_{a,n+2}y'(x)=D_a^{1/\rho_{n+1}}(A_{a,n+1}y'(x))=\varphi(x)$.

Denoting $A_{a,n+1}y'(x)\equiv Y(x)$ we can write $D_a^{1/\rho_{n+1}}Y(x)=\varphi(x)$. According to Lemma 1 (for $\lambda=0$), $Y(x)=-\frac{1}{\Gamma(1/\rho_{n+1})}\int_x^a(t-x)^{1/\rho_{n+1}-1}\varphi(t)dt$, i.e.

$$A_{a,n+1}y'(x)=-\frac{1}{\Gamma(1/\rho_{n+1})}\int_x^a(t-x)^{1/\rho_{n+1}-1}\varphi(t)dt. \tag{19}$$

Since Lemma 3 in true, we have

$$\begin{gathered} y(x)=\frac{(-1)^{n+1}}{\Gamma(1+\lambda_n)}\int_x^a(t-x)^{\lambda_n}dt\left(-\frac{1}{\Gamma(1/\rho_{n+1})}\int_t^a(\tau-t)^{1/\rho_{n+1}-1}\varphi(\tau)d\tau\right)dt= \\ =\frac{(-1)^{n+2}}{\Gamma(1+\lambda_n)\Gamma(1/\rho_{n+1})}\int_x^a\varphi(\tau)d\tau\int_x^\tau(t-x)^{\lambda_n}(\tau-t)^{1/\rho_{n+1}-1}dt. \end{gathered} \tag{20}$$

Obviously,

$$\begin{gathered} \int_x^\tau(t-x)^{\lambda_n}(\tau-t)^{1/\rho_{n+1}-1}dt=(\tau-x)^{\lambda_n+1/\rho_{n+1}}\int_0^1(1-\nu)^{\lambda_n}\nu^{1/\rho_{n+1}-1}d\nu= \\ =\frac{(\tau-x)^{\lambda_{n+1}}\Gamma(1+\lambda_n)\Gamma(1/\rho_{n+1})}{\Gamma(1+\lambda_{n+1})}. \end{gathered} \tag{21}$$

From (20) and (21) we get $y(x)=\frac{(-1)^{n+2}}{\Gamma(1+\lambda_{n+1})}\int_x^a(t-x)^{\lambda_{n+1}}\varphi(t)dt$, i.e. Lemma 3 is true for any $n\geq 1$.

Lemma 4. For any $n\geq 0$ the following relations hold:

$$1.\ A_{a,k}\left\{\frac{(a-x)^{\lambda_n}}{\Gamma(1+\lambda_n)}\right\}=A^*_{a,k}\left\{\frac{(a-x)^{\lambda_n}}{\Gamma(1+\lambda_n)}\right\}=0,\ k\geq n+1,\ x\in(0,a); \tag{22}$$

$$2.\ \ A^*_{a,n}\left\{\frac{(a-x)^{\lambda_n}}{\Gamma(1+\lambda_n)}\right\}\Bigg|_{x=a}=(-1)^n; \tag{23}$$

$$3.\ \ A^*_{a,k}\left\{\frac{(a-x)^{\lambda_n}}{\Gamma(1+\lambda_n)}\right\}\Bigg|_{x=a}=0,\ \ 0\leq k\leq n-1. \tag{24}$$

Lemma 4 is easy to prove similarly to the proof of Lemma 1 from [2] .

Lemma 5. For any $n\geq 0$ the coefficients $\{C_k\}_0^n$ of the sum

$$P_n(x)=\sum_{k=0}^n C_k\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}, \tag{25}$$

can be determined by the formulas

$$C_k=(-1)^kA^*_{a,k}\{P_n(x)\}\big|_{x=a},\ \ 0\leq k\leq n. \tag{26}$$

Proof. Assuming that $0 \le j \le n$. We apply the operator $A^*_{a,j}$ to the function $P_n(x)$. Then, using (22) and (24), we can have

$$\begin{aligned} A^*_{a,j}P_n(x) &= \sum_{k=0}^{n} C_k A^*_{a,j}\left\{\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}\right\} = \\ &= C_j A^*_{a,j}\left\{\frac{(a-x)^{\lambda_j}}{\Gamma(1+\lambda_j)}\right\} + \sum_{k=j+1}^{n} C_k A^*_{a,j}\left\{\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}\right\}. \end{aligned} \tag{27}$$

From (27) for $x=a$ we get $A^*_{a,j}\{P_n(x)\}\big|_{x=a} = (-1)^j C_j, \quad 0 \le j \le n,$ i.e. $C_j = (-1)^j A^*_{a,j}\{P_n(x)\}\Big|_{x=a}$. □

We denote by $C_{n+1}\{(0,a),\langle\rho_j\rangle\}$ the set of functions $f(x)$ satisfying the following conditions:

1) the functions $A^*_{a,k}f(x), k=0,1,\ldots,n$, are continuous on $[0,a]$;
2) the functions $A_{a,k}f(x), k=0,1,\ldots,n,n+1$, are continuous on $(0,a)$ and belongs to $L(0,a)$.

It is easy to see that each the function $\dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}$, $k=0,1,\ldots,$ and each polynomial $P_n(x) = \sum\limits_{k=0}^{n} C_k \dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}$ belongs to the class $C_{n+1}\{(0,a),\langle\rho_j\rangle\}$.

We prove the following theorem.

Theorem 1. If $f(x) \in C_{n+1}\{(0,a),\langle\rho_j\rangle\}$, then for any $n \ge 1$

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} (-1)^k A^*_{a,k} f(a) \frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)} + R_n(x), \\ R_n(x) &= \frac{(-1)^{n+1}}{\Gamma(1+\lambda_n)} \int_x^a (t-x)^{\lambda_n} A_{a,n+1} f(t)\,dt. \end{aligned} \tag{28}$$

Proof. We put $P_n(x) = \sum\limits_{k=0}^{n} (-1)^k A^*_{a,k} f(a) \dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}$ and $R_n(x) = f(x) - P_n(x)$. We notice that

$$A^*_{a,k}\{R_n(x)\}\big|_{x=a} = 0,\ k=0,1,2,\ldots,n;\ \ A_{a,n+1}\{R_n(x)\} = A_{a,n+1}f(t). \tag{29}$$

Since $A_{a,n+1}\{R_n(x)\} = \prod\limits_{j=1}^{n} D_a^{1/\rho_j} R'_n(x)$ and $D_a^{-\alpha_k} A_{a,k} R'_n(x)\big|_{x=a} = 0,\ k=0,1,\ldots,n,$ using Lemma 3, we get $R_n(x) = \dfrac{(-1)^{n+1}}{\Gamma(1+\lambda_n)} \displaystyle\int_x^a (t-x)^{\lambda_n} A_{a,n+1} f(t)\,dt$. □

Note that the Taylor–Maclaurin type formulae are obtained in the papers [3–8] by author and also in articles [9, 11].

3. $\langle\rho_j\rangle$ Completely Monotone Functions. We denote by $C_\infty\{(0,a),\langle\rho_j\rangle\}$, the set of functions satisfying $f(x) \in C_{n+1}\{(0,a),\langle\rho_j\rangle\}$ for any $n \ge 0$. We say a function $f(x)$ is $\langle\rho_j\rangle$ completely monotone, if

1. $f(x) \in C_\infty\{(0,a),\langle\rho_j\rangle\}$; (30)
2. $(-1)^n A_{a,n} f(x) \ge 0,\ \ A_{a,0} f \equiv f,\ \ n \ge 0,\ \ x \in (0,a)$. (31)

We denote by $C^*_\infty\{(0,a),\langle\rho_j\rangle\}$ the class of $\langle\rho_j\rangle$ completely monotone functions.

Theorem 2. Let $f(x) \in C^*_\infty\{(0,a),\langle\rho_j\rangle\}$ and $\sum_{j=1}^{\infty} 1/\rho_j = \lambda_\infty = +\infty$. Then

$$f(x) = \sum_{k=0}^{\infty}(-1)^k A^*_{a,k} f(a)\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}, \quad x \in (0,a]. \tag{32}$$

Proof. From (28) we have

$$f(x) = \sum_{k=0}^{n}(-1)^k A^*_{a,k} f(a)\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)} + R_n(x),$$

$$R_n(x) = \frac{(-1)^{n+1}}{\Gamma(1+\lambda_n)}\int_x^a (t-x)^{\lambda_n} A_{a,n+1} f(t)dt.$$

Observe, that if $(-1)^k A_{a,k} f(x) \geq 0$, then $(-1)^k A^*_{a,k} f(x) \geq 0$, $k = 0,1,\ldots,$ since by (14) we have $A^*_{a,k} f(x) \equiv D_a^{-\alpha_k} A_{a,k} f(x)$. Let x_0 be some fixed number and $0 < x_0 < x < t < a$. Then

$$\begin{aligned} 0 \leq R_n(x) &= \frac{1}{\Gamma(1+\lambda_n)}\int_x^a (t-x)^{\lambda_n}\{(-1)^{n+1}A_{a,n+1}f(t)\}dt \leq \\ &\leq \max_{x\leq t\leq a}\left(\frac{t-x}{t-x_0}\right)^{\lambda_n}\frac{1}{\Gamma(1+\lambda_n)}\int_{x_0}^a (t-x_0)^{\lambda_n}\{(-1)^{n+1}A_{a,n+1}f(t)\}dt. \end{aligned} \tag{33}$$

On the other hand,

$$\begin{aligned} f(x_0) &= \sum_{k=0}^{n}(-1)^k A^*_{a,k} f(a)\frac{(a-x_0)^{\lambda_k}}{\Gamma(1+\lambda_k)} + \\ &+ \frac{1}{\Gamma(1+\lambda_n)}\int_{x_0}^a (t-x_0)^{\lambda_n}\{(-1)^{n+1}A_{a,n+1}f(t)\}dt. \end{aligned} \tag{34}$$

From (34) it follows

$$R_n(x_0) = \frac{1}{\Gamma(1+\lambda_n)}\int_{x_0}^a (t-x_0)^{\lambda_n}\{(-1)^{n+1}A_{a,n+1}f(t)\}dt \leq f(x_0). \tag{35}$$

From (33) and (35) we can have

$$0 \leq R_n(x) \leq \max_{x\leq t\leq a}\left(\frac{t-x}{t-x_0}\right)^{\lambda_n} f(x_0). \tag{36}$$

Obviously $\max_{x\leq t\leq a}\left(\frac{t-x}{t-x_0}\right)^{\lambda_n} = \left(\frac{a-x}{a-x_0}\right)^{\lambda_n} \to 0$, as $\lambda_n \xrightarrow[n\to\infty]{} +\infty$, consequently, from (36) we get $\lim_{n\to\infty} R_n(x) = 0$, $\forall x \in (0,a]$, i.e. $f(x) = \sum_{k=0}^{\infty}(-1)^k A^*_{a,k} f(a)\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)}$.

Theorem 3. Let $f(x) \in C^*_\infty\{(0,a),\langle\rho_j\rangle\}$ and $\sum_{j=1}^{\infty} 1/\rho_j = \lambda_\infty < +\infty$.

Then there exists a function $\mu(x)$ that is non-increasing on $(0,a]$ and bounded on any interval $[a_1,a] \subset (0,a]$ such that $f(x)$ possesses the representation

$$f(x) = \sum_{k=0}^{\infty}(-1)^k A^*_{a,k} f(a)\frac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)} - \frac{1}{\Gamma(1+\lambda_\infty)}\int_x^a (t-x)^{\lambda_\infty} d\mu(t),\ x \in (0,a). \tag{37}$$

Proof. Denote

$$g_n(x) = \int_x^a (-1)^{n+1} A_{a,n+1} f(t)dt, \ \ n \geq 1, \ \ x \in (0,a]. \tag{38}$$

Let us show that the sequence $\{g_n(x)\}_1^\infty$ satisfies the conditions of Helly's first theorem. Namely, for $x \in (0,a]$ we have

$$0 \le g_n(x) \le M, \quad \bigvee_0^a (g_n) \le M, \quad n \ge 1. \tag{39}$$

Let $0 < x_1 < x_2 < x < a$. Since $\lim\limits_{n\to\infty} \dfrac{(x_2-x_1)^{\lambda_n}}{\Gamma(1+\lambda_n)} = \dfrac{(x_2-x_1)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}$, there exists number n_0 such that for $n > n_0$, $\dfrac{(x_2-x_1)^{\lambda_n}}{\Gamma(1+\lambda_n)} > \dfrac{1}{2}\cdot\dfrac{(x_2-x_1)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}$, and consequently

$$\begin{gathered} \frac{1}{2}\cdot\frac{(x_2-x_1)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)} \int_x^a \{(-1)^{n+1} A_{a,n+1} f(t)\} dt \le \\ \le \frac{(x_2-x_1)^{\lambda_n}}{\Gamma(1+\lambda_n)} \int_x^a \{(-1)^{n+1} A_{a,n+1} f(t)\} dt \le \\ \le \frac{1}{\Gamma(1+\lambda_n)} \int_{x_1}^a (t-x_1)^{\lambda_n} \{(-1)^{n+1} A_{a,n+1} f(t)\} dt. \end{gathered} \tag{40}$$

Observe that

$$\frac{1}{\Gamma(1+\lambda_n)} \int_{x_1}^a (t-x_1)^{\lambda_n} \{(-1)^{n+1} A_{a,n+1} f(t)\} dt = R_n(x_1) \le f(x_1). \tag{41}$$

From (40) and (41) we get $\dfrac{1}{2}\cdot\dfrac{x_2-x_1)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)} g_n(x) \le f(x_1)$, i.e.

$$g_n(x) \le \frac{2\Gamma(1+\lambda_\infty)}{(x_2-x_1)^{\lambda_\infty}} f(x_1), \quad 0 < x_1 < x_2 < x < a. \tag{42}$$

It is easy to see that $\bigvee_x^a (g_n) = g_n(x) - g_n(a) = g_n(x) \le \dfrac{2\Gamma(1+\lambda_\infty) f(x_1)}{(x_2-x_1)^{\lambda_\infty}}$ $(\forall n \ge 1,\ g_n(x)$ is non-increasing). Using Helly's first theorem, we conclude from (42) that there exists a subsequence $g_{n_k}(x)$ such that $\lim\limits_{n_k\to\infty} g_{n_k}(x) = \mu(x)$, $x \in (0,a]$. We note that

$$\begin{gathered} R_{n_k}(x) = \frac{1}{\Gamma(1+\lambda_{n_k})} \int_x^a (t-x)^{\lambda_{n_k}} \{(-1)^{n_k+1} A_{a,n_k+1} f(t)\} dt = \\ = -\frac{1}{\Gamma(1+\lambda_{n_k})} \int_x^a (t-x)^{\lambda_{n_k}} d\left(\int_t^a (-1)^{n_k+1} A_{a,n_k+1} f(\tau) d\tau\right) = \\ = -\frac{1}{\Gamma(1+\lambda_{n_k})} \int_x^a (t-x)^{\lambda_{n_k}} d\{g_{n_k}(t)\}. \end{gathered} \tag{43}$$

From here, using Helly's second theorem, we get

$$\lim_{n_k\to\infty} R_{n_k}(x) = -\frac{1}{\Gamma(1+\lambda_\infty)} \int_x^a (t-x)^{\lambda_\infty} d\mu(t),$$

hence $f(x) = \sum\limits_{k=0}^{\infty} (-1)^k A_{a,k}^* f(a) \dfrac{(a-x)^{\lambda_k}}{\Gamma(1+\lambda_k)} - \dfrac{1}{\Gamma(1+\lambda_\infty)} \displaystyle\int_x^a (t-x)^{\lambda_\infty} d\mu(t)$. $\square$

In conclusion, consider the following example.

Let $f(x) = \dfrac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}$, $x \in [0,a]$. We showed that $f(x) \in C_\infty^*\{(0,a), \langle \rho_j \rangle\}$. Easy to get

$$A_{a,k}\left\{\frac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}\right\}=\frac{(-1)^k(a-x)^{\lambda_\infty-\sum\limits_{j=1}^{k-1}1/\rho_j-1}}{\Gamma\left(\lambda_\infty-\sum\limits_{j=1}^{k-1}1/\rho_j\right)}, \tag{44}$$

$$(-1)^k A_{a,k}\left\{\frac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}\right\}\geq,\ k=0,1,\ldots, \tag{44'}$$

$$(-1)^k A^*_{a,k}\left\{\frac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}\right\}=\frac{(a-x)^{\lambda_\infty-\sum\limits_{j=1}^{k}1/\rho_j}}{\Gamma\left(\lambda_\infty-\sum\limits_{j=1}^{k}1/\rho_j+1\right)},\ x\in[0,a]. \tag{45}$$

$$A^*_{a,k}\left\{\frac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}\right\}\Bigg|_{x=a}=0,\ k=0,1,\ldots \tag{46}$$

From (37) we get $f(x)=\dfrac{(a-x)^{\lambda_\infty}}{\Gamma(1+\lambda_\infty)}=-\dfrac{1}{\Gamma(1+\lambda_\infty)}\displaystyle\int_x^\infty (t-x)^{\lambda_\infty}d\mu(t)$, where $\mu(t)=\begin{cases}2, & 0\leq t<a,\\ 1, & t=a.\end{cases}$

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