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Mathematics

THE REPRESENTATION OF FUNCTIONS BY WALSH DOUBLE SYSTEM IN WEIGHTED $L^p_\mu[0,1)^2$-SPACES

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In this work we construct a weighted space L^p_μ, $p \geq 1$, in which functions with the norm of that space are presented by Walsh double series, which coefficients are monotone in all ways.

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Introduction. Let $|E|$ be the Lebesgue measure of a measurable set $E \subseteq [0,1)$ (or $E \subseteq [0,1) \times [0,1) = [0,1)^2$), and let $L^p[0,1)$, $p \geq 1$, be the class of all those measurable functions $f(x)$ on $[0,1)$ such that

$$\int_0^1 |f(x)|^p dx < \infty. \tag{1}$$

Let $\mu(x,y)$ be a positive Lebesgue-measurable function (weight function) defined on $[0,1)^2$. We denote by $L^p_\mu[0,1)^2$ the space of all measurable functions on $[0,1)^2$ with the norm

$$\|.\|_{L^p_\mu} = \left(\int_0^1 \int_0^1 |.|^p \mu(x,y) dxdy \right)^{1/p} < \infty : \ p \in [1,\infty). \tag{2}$$

In the sequel we will accept the terms "measure" and "measurable" in the sense of Lebesgue.

Definition 1. *The nonzero members of a double sequence* $\{b_{k,s}\}_{k,s=0}^{\infty}$ *are said to be in a monotonically decreasing order over all rays, if* $b_{k_2,s_2} < b_{k_1,s_1}$ *when* $k_2 \geq k_1$, $s_2 \geq s_1$, $k_2 + s_2 > k_1 + s_1$ $(b_{k_i,s_i} \neq 0,\ i = 1,2)$.

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Let $f(x,y) \in L^p[0,1)^2,\ p \geq 1$, and let

$$\sum_{k,s=0}^{\infty} c_{k,s}\varphi_k(x)\varphi_s(y) \tag{3}$$

be series with double Walsh system.

The spherical and rectangular partial sums of the series (3) will be denoted by $S_R(x,y) = \sum_{k^2+s^2\leq R^2} c_{k,s}\varphi_k(x)\varphi_s(y)$ and $S_{N,M}(x,y) = \sum_{k=0}^{N}\sum_{s=0}^{M} c_{k,s}\varphi_k(x)\varphi_s(y)$, respectively.

Definition 2. *Let $f(x,y) \in L^p_\mu[0,1)^2$. We will say that the series (3) converges to the function $f(x,y)$ in $L^p_\mu[0,1)^2$-norm with respect to spheres, if*

$$\lim_{R\to\infty}\left(\int_0^1\int_0^1 |S_R(x,y)-f(x,y)|^p \mu(x,y)dxdy\right)^{1/p} = 0.$$

The convergence with respect to rectangles is defined in the same way. More general statements of these definitions can be found in [1–12].

Definition 3. *A series $\sum_{k,s=0}^{\infty} b_{k,s}\varphi_k(x)\varphi_s(y)$ is called universal in $L^p_\mu[0,1)^2$ with respect to the subseries, if for every function $f(x,y) \in L^p_\mu[0,1)^2$ there exists a subseries $\sum_{i,j=0}^{\infty} b_{k_i,s_j}\varphi_{k_i}(x)\varphi_{s_j}(y)$, which converges to f in $L^p_\mu[0,1)^2$-norm.*

In this work we will discuss the existence of Walsh universal double series with respect to the subseries in weighted $L^p_\mu[0,1)^2$-spaces.

Note that different kind of partial sums (e.g. spherical, rectangular, square) behave differently in the concepts of convergence in $L^p[0,1)^2, p \geq 1$, and convergence almost everywhere. Also, many classical results (for instance, Carleson's [2], Riesz's [13] and Kolmogorov's [14] theorems) cannot be extended from the one-dimensional case to the two-dimensional (see [3, 15], [16]).

In [14] Harris constructed a function $f \in L^p[0,1)^2$ with $1 \leq p < 2$ such that the Fourier–Walsh series of $f(x,y)$ in the Walsh double system diverges almost everywhere and in $L^p[0,1)^2$-norm with respect to spheres.

Thus for a given function $f(x,y) \in L^p[0,1)^2$ it is impossible to find a double series in the Walsh double system converging to $f(x,y)$ either in $L^p[0,1)^2$-norm or almost everywhere with respect to spheres.

In the present work we prove that for any $\varepsilon > 0$ there exists a measurable set $E \subset [0,1)^2$ with $|E| > 1-\varepsilon$ such that for any function $f(x,y) \in L^p(E),\ p \geq 1$, one can find a series $\sum_{k,s=0}^{\infty} b_{k,s}\varphi_k(x)\varphi_s(y)$ with respect to the Walsh double system, which converges to the function $f(x,y)$ in the $L^p(E)$-norm with respect to spheres, that is

$$\lim_{R\to\infty}\int\int_E \left|\sum_{k^2+s^2\leq R^2} b_{k,s}\varphi_k(x)\varphi_s(y) - f(x,y)\right|^p dxdy = 0.$$

The following theorem is true:

Theorem 1. *$\forall\varepsilon > 0$ there exist a set $E \subset [0,1)^2$ with $|E| > 1-\varepsilon$ and a measurable (weight) function $\mu(x,y) : 0 < \mu(x,y) \le 1, (x,y) \in [0,1)^2$, with $\mu(x,y) = 1$ on E such that for each $p \in [1,\infty)$ and for every function $f(x,y) \in L^p_\mu[0,1)^2$ there exists a series with the following property:*

$$\lim_{R\to\infty}\int_0^1\int_0^1\left|\sum_{k^2+s^2\le R^2} b_{k,s}\varphi_k(x)\varphi_s(y) - f(x,y)\right|^p \mu(x,y)dxdy = 0.$$

This stronger theorem follows from Theorem 1:

Theorem 2. *For $\forall\varepsilon > 0$ there exist a set $E \subset [0,1)^2$, $|E| > 1-\varepsilon$, a measurable (weight) function $\mu(x,y) : 0 < \mu(x,y) \le 1, (x,y) \in [0,1)^2$, with $\mu(x,y) = 1$ on E, a series of the form $\sum_{k,s=0}^{\infty} d_{k,s}\varphi_k(x)\varphi_s(y)$, where $\sum_{k,s=0}^{\infty}|d_{k,s}|^r < \infty$ for all $r > 2$ and non-zero terms in $\left\{|d_{k,s}|\right\}_{k,s=0}^{\infty}$ are in the decreasing order over all rays, such that for each $p \in [1,\infty)$ and for every function $f(x,y) \in L^p_\mu[0,1)^2$ one can find numbers $\delta_{k,s} = 0$ or 1 such that*

$$\lim_{R\to\infty}\int_0^1\int_0^1\left|\sum_{k^2+s^2\le R^2} \delta_{k,s}d_{k,s}\varphi_k(x)\varphi_s(y) - f(x,y)\right|^p \mu(x,y)dxdy = 0.$$

Remark. Observe that one can not claim $\mu(x,y) \equiv 1$ in Theorem 2. It can be easily shown that the assumption of the existence of such universal series $\sum_{k,s=0}^{\infty} c_{k,s}\varphi_k(x)\varphi_s(y)$ with respect to the subseries for the space $L^p[0,1)^2, p \ge 1$, simply leads to contradiction. Indeed, if that assumption is true, then for the function $f(x,y) = 5c_{k_0,s_0}\varphi_{k_0}(x)\varphi_{s_0}(y)$, where $k_0, s_0 > 1$ are any natural numbers and $c_{k_0,s_0} \ne 0$, one can find numbers $\delta_k = 0$ or 1 such that

$$\lim_{m\to\infty}\int_0^1\int_0^1\left|\sum_{k,s=0}^{m} \delta_{k,s}c_{k,s}\varphi_k(x)\varphi_s(y) - 5c_{k_0,s_0}\varphi_{k_0}(x)\varphi_{s_0}(y)\right| dxdy = 0.$$

Hence, we will simply get $\delta_{k_0,s_0} = 5 > 1$.

The Main Lemma. The Walsh system is defined as follows. Let $r(x)$ be a 1-periodic function on $[0,1)$ defined by $r = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, where $\chi_E(x)$ denotes the characteristic function of the set E, that is,

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

The Rademacher system $R = \{r_n : n = 0,1,\dots\}$ is defined by

$$r_n(x) = r(2^n x) \text{ for all } x \in R,\ n = 0,1,\dots \tag{4}$$

Recall the definition of the Walsh system $\{\varphi_n\}(x)$ in Paley order (see [13]). Define

$$\varphi_n(x) = \prod_{k=0}^{\infty} r_k^{n_k}(x), \tag{5}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n with n_k 0 or 1.

The following lemma, which immediately follows from Lemma 4 from [17], plays a central role in the proof of our Theorem:

Lemma 1. *Let $\{\varphi_k\}$ be the Walsh system. Then for each $0<\delta<1$ there exists a measurable positive function $\mu(x,y)$ with $\left|\{(x,y)\in[0,1)^2;\mu(x,y)=1\}\right|>1-\delta$ such that for any numbers $\varepsilon\in(0,1)$, $N\in\mathbb{N}$, $p_0>1$ and for each function $f\in L^{p_0}[0,1)^2, \|f\|_{p_0}>0$, one can find a polynomial $Q(x,y)$ of the form*

$$Q(x,y)=\sum_{k,s=N}^{M} c_{k,s}\varphi_k(x)\varphi_s(y),$$

satisfying the following conditions:

1) *the nonzero coefficients in $\{|c_{k,n}|,\ k,n=N,\dots,M\}$ are in decreasing order over all rays;*

2) $$\sum_{k,n=N}^{M}|c_{k,n}|^{2+\varepsilon}<\varepsilon;$$

3) $$\int_0^1\int_0^1|Q(x,y)-f(x,y)|^{p_0}\mu(x,y)dxdy<\varepsilon^{p_0};$$

4) $$\max_{\sqrt{2}N\le R\le\sqrt{2}M}\left(\int_0^1\int_0^1\left|\sum_{2N^2\le k^2+s^2\le R^2}c_{k,s}\varphi_k(x)\varphi_s(y)\right|^p\mu(x,y)dxdy\right)^{1/p}\le$$
$$\le\left(\left(\int_0^1\int_0^1|f(x,y)|^p\mu(x,y)dxdy\right)^{1/p}+\varepsilon\right)\ for\ all\ p\in[1,p_0].$$

Proof of Theorem 2.

Proof. Let $0<\varepsilon<1,\ p_n\nearrow\infty\ (p_1>1)$ and let

$$\left\{f_k(x,y)\right\}_{k=1}^{\infty}\tag{6}$$

be a sequence of all polynomials in the Walsh system with rational coefficients.

Successively applying Lemma 1, we can find a measurable weight function $\mu(x,y)$, a set $E\subset[0,1)^2$ such that

$$\mu(x,y)=1 \text{ on } E,\ |E|>1-\varepsilon,\tag{7}$$

and polynomials

$$\overline{Q}_n(x,y)=\sum_{k,s=m_{n-1}}^{m_n-1}b_{k,s}^{(n)}\varphi_k(x)\varphi_s(y), m_n\nearrow,\tag{8}$$

which satisfy the following conditions for every $n\ge1$:

$$\int_0^1\int_0^1 \left|\overline{Q}_n(x,y)-f_n(x,y)\right|^{p_n}\mu(x,y)dxdy \le 2^{-8p_n(n+1)}. \tag{9}$$

All nonzero members in the sequence $\left\{\left|b_{k,s}^{(n)}\right| \ k,s\in[m_{n-1},m_n)\right\}$ are in decreasing order over all rays for any fixed $n\ge 1$ and

$$\max_{k,s\in[m_{n-1},m_n)}\left|b_{k,s}^{(n)}\right| < \min_{(k,s)\in spec\overline{Q}_{n-1}}\left|b_{k,s}^{(n-1)}\right| \text{ for all } n=1,2\ldots, \tag{10}$$

$$\sum_{k,s=m_{n-1}}^{m_n-1}\left|b_{k,s}^{(n)}\right|^{2+2^{-n}} < \frac{1}{2^{8(n+1)}},\ n\ge 1, \tag{11}$$

$$\begin{gathered}\max_{\sqrt{2}m_{n-1}\le R<\sqrt{2}m_n}\left(\int_0^1\int_0^1\left|\sum_{2m_{n-1}^2\le k^2+s^2\le R^2} b_{k,s}^{(n)}\varphi_k(x)\varphi_s(y)\right|^p\mu(x,y)dxdy\right)^{1/p}\le\\ \le 2\left(\int_0^1\int_0^1|f_n(x,y)|^p\mu(x,y)dxdy\right)^{1/p}+2^{-2n} \text{ for all } p\in[1,p_n].\end{gathered} \tag{12}$$

We put

$$b_{k,s}=\begin{cases} b_{k,s}^{(n)}, & k,s\in[m_{n-1},m_n),\ n\ge 1,\\ 0, & \text{in other cases.}\end{cases} \tag{13}$$

□

Let $f(x,y)\in L_\mu^p[0,1)^2, \forall p\ge 1$. Now assume that the polynomials

$$\overline{Q}_{l_j}(x,y)=\sum_{k,s=m_{l_{j-1}}}^{m_{l_j}-1} b_{k,s}^{(l_j)}\varphi_k(x)\varphi_s(y),\ 1\le j\le q-1, \tag{14}$$

have been defined satisfying the conditions

$$\int_0^1\int_0^1\left|f(x,y)-\sum_{j=1}^{q'}\overline{Q}_{l_j}(x,y)\right|^p\mu(x,y)dxdy<2^{-2q'},\ 1\le q'\le q-1, \tag{15}$$

$$\max_{\sqrt{2}m_{l_{j-1}}\le R<\sqrt{2}m_{l_j}}\int_0^1\int_0^1\left|\sum_{2m_{l_{j-1}}^2\le k^2+s^2\le R^2} b_{k,s}^{(l_j)}\varphi_k(x)\varphi_s(y)\right|^p\mu(x,y)dxdy<2^{-l_j\cdot p}. \tag{16}$$

Choose the function f_{l_q} from the sequence F (see (6)) such that

$$\left(\int_0^1\int_0^1\left|f_{l_q}(x,y)-\left[f(x,y)-\sum_{j=1}^{q-1}\overline{Q}_{l_j}(x,y)\right]\right|^p\mu(x,y)dxdy\right)^{1/p}<2^{-2(q+2)}. \quad (17)$$

It follows from (15) and (17) that

$$\left(\int_0^1\int_0^1\left|f_{l_q}(x,y)\right|^p\mu(x,y)dxdy\right)^{1/p}<2^{-2(q-1)}+2^{-2(q+2)}. \quad (18)$$

Taking into account (12) and (16)–(18), we have

$$\begin{gathered}\left(\int_0^1\int_0^1\left|f(x,y)-\sum_{j=1}^{q}\overline{Q}_{l_j}(x,y)\right|^p\mu(x,y)dxdy\right)^{1/p}\le\\ \le\left(\int_0^1\int_0^1\left|\overline{Q}_{l_q}(x,y)-f_{l_q}(x,y)\right|^p\mu(x,y)dxdy)\right)^{1/p}+\\ +\left(\int_0^1\int_0^1\left|f_{l_q}(x,y)-\left[f(x,y)-\sum_{j=1}^{q-1}\overline{Q}_{l_j}(x,y)\right]\right|^p\mu(x,y)dxdy)\right)^{1/p}\le\\ \le 2^{-8l_q}+2^{-2(q+2)}<2^{-2q},\end{gathered} \quad (19)$$

$$\max_{\sqrt{2}m_{l_q-1}\le R<\sqrt{2}m_{l_q}}\int_0^1\int_0^1\left|\sum_{2m_{l_q-1}^2\le k^2+s^2\le R^2}b_{k,s}^{(l_q)}\varphi_k(x)\varphi_s(y)\right|^p\mu(x,y)dxdy<2^{-l_q\cdot p}. \quad (20)$$

It is clear that we can define by induction polynomials

$$\overline{Q}_{l_q}(x,y)=\sum_{k,s=m_{l_q-1}}^{m_{l_q}-1}b_{k,s}^{(l_q)}\varphi_k(x)\varphi_s(y), \quad (21)$$

satisfying conditions (15) and (16) for all $q\ge 1$. We set

$$\delta_{k,s}=\begin{cases}1, & k,s\in\bigcup_{q=1}^{\infty}[m_{l_q-1},m_{l_q}),\\ 0, & \text{in other cases.}\end{cases} \quad (22)$$

By (19)–(22) we have

$$\lim_{R\to\infty}\left(\int_0^1\int_0^1\left|\sum_{0\le k^2+s^2\le R^2}\delta_{k,s}b_{k,s}\varphi_k(x)\varphi_s(y)-f(x,y)\right|^p\mu(x,y)dxdy\right)^{1/p}=0, \quad (23)$$

i. e. the Theorem 2 is proved.

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Լ. Ս. ՍԻՄՈՆՅԱՆ

ՖՈՒՆԿՑԻԱՆՆԵՐԻ ՆԵՐԿԱՅԱՑՈՒՄՆ ՈՒՈԼՇԻ ԿՐԿՆԱԿԻ ՀԱՄԱԿԱՐԳՈՎ $L^p_\mu[0,1)^2$ ԿՇՌԱՅԻՆ ՏԱՐԱԾՈՒԹՅՈՒՆՆԵՐՈՒՄ

Այս աշխատանքում կառուցվում է L^p_μ, $p \geq 1$, քշռային տարածություն, որի ֆունկցիաներն այդ տարածության նորմով ներկայացվում են բոլոր ուղղություններով մոնոտոն գործակիցներ ունեցող Ուոլշի կրկնակի շարքերով: