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M a t h e m a t i c s

ON AN INTEGRO-DIFFERENTIAL EQUATION OF PSEUDOPARABOLIC-PSEUDOHYPERBOLIC TYPE WITH DEGENERATE KERNELS

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In the article the questions of solvability of boundary value problem for a homogeneous pseudoparabolic-pseudohyperbolic type integro-differential equation with degenerate kernels are considered. The Fourier method based on separation of variables is used. A criterion for the one-valued solvability of the considering problem is found. Under this criterion the one-valued solvability of the problem is proved.

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Problem Statement. The partial differential equations of third and fourth order are important with their physical applications [1–5]. Problems, where the type of differential equation is changing in the considering domain, have important applications [6–8]. The mixed type differential equations have been studied by many authors, in particular in [9–16].

In the present paper we consider the one-valued solvability of nonlocal problem for a mixed type integro-differential equation with degenerate kernels. So, in the rectangular domain $\Omega = \{(t,x) | -T < t < T, \quad 0 < x < l\}$ we consider the following mixed type equation:

$$\begin{cases} U_t - U_{txx} - U_{xx} + \nu \displaystyle\int_0^T K_1(t,s) U(s,x) ds = 0, \quad t > 0, \\ U_{tt} - U_{ttxx} - U_{xx} + \nu \displaystyle\int_{-T}^0 K_2(t,s) U(s,x) ds = 0, \quad t < 0, \end{cases} \tag{1}$$

where T and l are given positive real numbers; ν is spectral real parameter, $K_j(t,s) = a_j(t)\, b_j(s), \;\; a_j(t),\, b_j(s) \in C[-T;\, T], \;\; j = 1,2.$

Problem. Find in the domain Ω the function

$$U(t,x) \in C(\overline{\Omega}) \cap C^1(\Omega \cup \{x=0\} \cup \{x=l\}) \cap C^{1,2}(\Omega_+) \cap C^{2,2}(\Omega_-) \cap$$
$$\cap C^{1+2}_{t,x}(\Omega_+) \cap C^{2+2}_{t,x}(\Omega_-),$$

satisfying to the Eq. (1) and following conditions:

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$$\int_0^T U(t,x)dt = \varphi(x), \quad 0 \le x \le l, \tag{2}$$

$$U(t,0) = U(t,l) = 0, \quad -T \le t \le T, \tag{3}$$

where C^r is a class of functions having continuous derivatives $\frac{\partial^r}{\partial t^r}$, $\frac{\partial^r}{\partial x^r}$; $C^{r,s}$ is a class of functions having continuous derivatives $\frac{\partial^r}{\partial t^r}$, $\frac{\partial^s}{\partial x^s}$; $C^{r+s}_{t,x}$ is a class of functions having continuous derivatives $\frac{\partial^{r+s}}{\partial t^r \partial x^s}$, $r = \overline{1,r_0}$, $s = \overline{1,s_0}$, $r \le r_0$, $s \le s_0$ are arbitrary natural numbers, $\varphi(x)$ is given sufficiently smooth function, $\varphi(0) = \varphi(l) = 0, \Omega_- = \{(t,x)| -T < t < 0,\ 0 < x < l\}$, $\Omega_+ = \{(t,x)|0 < t < T,\ 0 < x < l\}$, $\overline{\Omega} = \{(t,x)| -T \le t \le T, 0 \le x \le l\}$.

A Formal Solution of the Boundary Value Problem (1)–(3). Solution of the Eq. (1) in domain Ω is sought in the form of the following Fourier series:

$$U(t,x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} u_n(t) \sin \mu_n x, \tag{4}$$

where the functions $\vartheta_n(x) = \sqrt{\frac{2}{l}} \sin \mu_n x$ as the eigenfunctions of the spectral problem $\vartheta''(x) + \mu^2 \vartheta(x) = 0$, $\vartheta(0) = \vartheta(l) = 0$, $0 < \mu$ and form a complete system of orthonormal functions in $L_2[0;\ l]$, while $\mu_n = \frac{\pi n}{l}$ are the corresponding eigenvalues and

$$u_n(t) = \sqrt{\frac{2}{l}} \int_0^l U(t,x) \sin \mu_n x dx,\ n = 1,2,\dots \tag{5}$$

Substituting the series (4) into Eq. (1), we obtain

$$u_n'(t) + \lambda_n^2 u_n(t) = \nu \int_0^T a_1(t) b_1(s) u_n(s) ds, \quad t > 0, \tag{6}$$

$$u_n''(t) + \lambda_n^2 u_n(t) = \nu \int_{-T}^0 a_2(t) b_2(s) u_n(s) ds, \quad t < 0, \tag{7}$$

where $\lambda_n^2 = \frac{\mu_n^2}{1+\mu_n^2}$, $\mu_n = \frac{\pi n}{l}$.

By denoting

$$\alpha_n = \int_0^T b_1(s) u_n(s) ds, \tag{8}$$

$$\beta_n = \int_{-T}^0 b_2(s) u_n(s) ds, \tag{9}$$

the Eqs. (6) and (7) can be written by

$$u_n'(t) + \lambda_n^2 u_n(t) = \nu a_1(t) \alpha_n,\ t > 0, \quad u_n''(t) + \lambda_n^2 u_n(t) = \nu a_2(t) \beta_n,\ t < 0. \tag{10}$$

We solve Eqs. (10) by the method of variation of arbitrary constants

$$u_n(t) = A_n \exp\{-\lambda_n^2 t\} + \eta_{1n}(t),\ t > 0, \tag{11}$$

$$u_n(t) = C_n \cos \lambda_n t + D_n \sin \lambda_n t + \eta_{2n}(t),\ t < 0, \tag{12}$$

where A_n, B_n, C_n are while arbitrary constants to be determined and $\eta_{1n}(t) = \nu \alpha_n h_n(t)$, $\quad \eta_{2n}(t) = \nu \beta_n \delta_n(t)$,

$$h_n(t)=\int_0^t \exp\big\{-\lambda_n^2(t-s)\big\}a_1(s)ds, \quad \delta_n(t)=\frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-s)a_2(s)ds.$$

From the statement of the problem it follows that $U(0+0,x)=U(0-0,x)$; $U_t(0+0,x)=U_t(0-0,x)$. Hence, taking into account (5), we obtain

$$\begin{aligned} u_n(0+0)&=\sqrt{\frac{2}{l}}\int_0^l U(0+0,x)\sin\mu_n x dx=\\ &=\sqrt{\frac{2}{l}}\int_0^l U(0-0,x)\sin\mu_n x dx=u_n(0-0). \end{aligned} \tag{13}$$

Differentiating (5) one times with respect to t similarly (13), we derive

$$\begin{aligned} u_n'(0+0)&=\sqrt{\frac{2}{l}}\int_0^l U_t(0+0,x)\sin\mu_n x dx=\\ &=\sqrt{\frac{2}{l}}\int_0^l U_t(0-0,x)\sin\mu_n x dx=u_n'(0-0). \end{aligned} \tag{14}$$

From (11) and (12), taking into account (13) and (14), we obtain that $B_n=A_n$, $C_n=-\lambda_n A_n$. Then functions (11) and (12) take the form

$$u_n(t)=A_n\exp\big\{-\lambda_n^2 t\big\}+\eta_{1n}(t),\ \ t>0, \tag{15}$$

$$u_n(t)=A_n\cos\lambda_n t-\lambda_n A_n\sin\lambda_n t+\eta_{2n}(t),\ \ t<0. \tag{16}$$

Taking into account (5), the condition (2) takes the following form

$$\int_0^T u_n(t)dt=\sqrt{\frac{2}{l}}\int_0^l\int_0^T U(t,x)dt\sin\mu_n x dx=\sqrt{\frac{2}{l}}\int_0^l \varphi(x)\sin\mu_n x dx=\varphi_n, \tag{17}$$

where $\varphi_n=\sqrt{\frac{2}{l}}\int_0^l \varphi(x)\sin\mu_n x dx,\ \ n=1,2,\ldots$

To find the unknown coefficients A_n in (15) and (16), we use the condition (17)

$$\begin{aligned} \int_0^T u_n(t)dt&=\int_0^T\Big[A_n\exp\big\{-\lambda_n^2 t\big\}+\eta_{1n}(t)\Big]dt=\\ &=-\frac{A_n}{\lambda_n^2}\Big[\exp\big\{-\lambda_n^2 T\big\}-1\Big]+\xi_{1n}(t)=\varphi_n, \end{aligned} \tag{18}$$

where $\xi_{1n}=\int_0^T \eta_{1n}(t)dt$.

Since $0<T<\infty,\ 0<\lambda_n^2<1$, we have $\exp\big\{-\lambda_n^2 T\big\}\neq 1$. So from Eq. (18) A_n is uniquely determined by

$$A_n=\frac{\lambda_n^2}{\sigma_n}\Big(\varphi_n-\xi_{1n}\Big), \quad \text{where} \quad \sigma_n=1-\exp\big\{-\lambda_n^2 T\big\}.$$

Substituting the founded values A_n into formulas (15) and (16), we get

$$u_n(t)=\frac{\lambda_n^2}{\sigma_n}(\varphi_n-\xi_{1n})\exp\big\{-\lambda_n^2 t\big\}+\eta_{1n}(t),\ \ t>0, \tag{19}$$

$$u_n(t)=\frac{\lambda_n^2}{\sigma_n}(\varphi_n-\xi_{1n})\left[\cos\lambda_n t-\lambda_n\sin\lambda_n t\right]+\eta_{2n}(t),\ \ t<0. \tag{20}$$

Taking into account that $\xi_{1n} = \int_0^T \eta_{1n}(t)dt$, $\eta_{1n}(t) = \nu\alpha_n h_n(t)$ and $\eta_{2n}(t) = \nu\beta_n\delta_n(t)$, we rewrite the formulas (19) and (20) as follows

$$u_n(t) = \varphi_n M_{1n}(t) + \nu\alpha_n M_{2n}(t), \ \ t > 0, \tag{21}$$

$$u_n(t) = \varphi_n N_{1n}(t) - \nu\alpha_n N_{2n}(t) + \nu\beta_n\delta_n(t), \ \ t < 0, \tag{22}$$

where

$$M_{1n}(t) = \frac{\lambda_n^2}{\sigma_n}\exp\left\{-\lambda_n^2 t\right\}, \quad M_{2n}(t) = h_n(t) - M_{1n}(t)\int_0^T h_n(t)dt,$$

$$N_{1n}(t) = \frac{\lambda_n^2}{\sigma_n}\left[\cos\lambda_n t - \lambda_n \sin\lambda_n t\right], \quad N_{2n}(t) = N_{1n}(t)\int_0^T h_n(t)dt,$$

$$0 < \lambda_n = \sqrt{\frac{\mu_n^2}{1+\mu_n^2}} < 1, \quad \mu_n = \frac{\pi n}{l}.$$

Substituting (21) into (9) and (22) into (10), we obtain the countable system of two algebraic equations (CSTAE) of variables α_n and β_n

$$\begin{cases} \alpha_n\left(1-\nu P_{2n}\right) = \varphi_n P_{1n}, \\ \alpha_n \nu Q_{2n} + \beta_n\left(1-\nu Q_{3n}\right) = \varphi_n Q_{1n}, \end{cases} \tag{23}$$

where $P_{1n} = \int_0^T b_1(s)M_{1n}(s)ds$, $P_{2n} = \int_0^T b_1(s)M_{2n}(s)ds$, $Q_{1n} = \int_{-T}^0 b_2(s)N_{1n}(s)ds$, $Q_{2n} = \int_{-T}^0 b_2(s)N_{2n}(s)ds$, $Q_{3n} = \int_{-T}^0 b_2(s)\delta_n(s)ds$.

For solvability of CSTAE (23) we impose the following condition

$$\nu = \nu_n \neq \frac{1}{P_{2n}}, \quad \nu = \nu_n \neq \frac{1}{Q_{3n}}. \tag{24}$$

We subtract the values $\nu_{1n} = \frac{1}{P_{2n}}$ and $\nu_{2n} = \frac{1}{Q_{3n}}$ of spectral parameter ν from the set of real numbers $R = (-\infty;\infty)$. The obtained set $\Lambda = R\setminus\{\nu_1,\nu_2\}$ is called the set of regular values of the parameter ν. For all values $\nu \in \Lambda$ the condition (24) is fulfilled. For regular values of the kernel of the mixed integro-differential Eq. (1), first we solve CSTAE (23), and then problem (1)–(3). Substituting the solution of CSTAE (23)

$$\alpha_n = \varphi_n\frac{P_{1n}}{1-\nu P_{2n}}, \quad \beta_n = \varphi_n\left[\frac{Q_{1n}}{1-\nu Q_{3n}} - \frac{Q_{2n}}{1-\nu Q_{3n}}\cdot\frac{P_{1n}}{1-\nu P_{2n}}\right]$$

in (21) and (22), we derive

$$u_n(t,\nu) = \varphi_n\Phi_n(t,\nu), \ \ t > 0, \tag{25}$$

$$u_n(t,\nu) = \varphi_n\Psi_n(t,\nu), \ \ t < 0, \tag{26}$$

where

$$\Phi_n(t,\nu) = M_{1n}(t) + \nu M_{2n}(t)\frac{P_{1n}}{1-\nu P_{2n}},$$

$$\Psi_n(t,\nu) = N_{1n}(t) - \nu N_{2n}(t)\frac{P_{1n}}{1-\nu P_{2n}} + \nu\delta_n(t)\left[\frac{Q_{1n}}{1-\nu Q_{3n}} - \frac{Q_{2n}}{1-\nu Q_{3n}}\cdot\frac{P_{1n}}{1-\nu P_{2n}}\right].$$

Now we substitute (25) and (26) into series (4) and we get

$$U(t,x,\nu) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\varphi_n\Phi_n(t,\nu)\sin\mu_n x,\ \ t>0, \tag{27}$$

$$U(t,x,\nu) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\varphi_n\Psi_n(t,\nu)\sin\mu_n x,\ \ t<0. \tag{28}$$

The Justification of the Solvability of the Boundary Value Problem (1)–(3). We show that under certain conditions with respect to the function $\varphi(x)$ the series (27) and (28) converge absolutely and uniformly. Indeed, in the formulation of the problem the functions $M_{1n}(t)$, $M_{2n}(t)$ are uniformly bounded on the segment $[0;\ T]$, and the functions $N_{1n}(t)$, $N_{2n}(t)$ and $\delta_n(t)$ are uniformly bounded on the segment $[-T;0]$. We consider such regular values of the spectral parameter $\nu\in\Lambda$, for which $\left|\Phi_n(t,\nu)\right|<\infty$ for all $t\in[0;\ T]$ and $\left|\Psi_n(t,\nu)\right|<\infty$ for all $t\in[-T;0]$. We note that $0<\lambda_n<1$. So for any natural n from the (25) and (26) we have estimates

$$\left|u_n(t)\right| \le C_{1n}\left|\varphi_n\right|, \tag{29}$$

$$\left|u_n''(t)\right| \le C_{2n}\varphi_n\big|, \tag{30}$$

where

$$C_{1n} = \max\left\{\max_{t\in[0,\ T]}\left|\Phi_n(t,\ \nu)\right|;\ \max_{t\in[-T,\ 0]}\left|\Psi_n(t,\nu)\right|\right\},$$

$$C_{2n} = \max\left\{\max_{t\in[0,\ T]}\left|\Phi_n''(t,\nu)\right|;\ \max_{t\in[-T,\ 0]}\left|\Psi_n''(t,\nu)\right|\right\}.$$

Condition A. Suppose that the following condition is satisfied:

$$\left(\sum_{n=1}^{\infty}\left|C_{in}\right|^2\right)^{1/2} < \infty,\ \ i=1,2.$$

Condition B. Suppose that the function $\varphi(x)\in C^2[0;l]$ on the segment $[0;l]$ has piecewise-continuous third-order derivatives and $\varphi(0)=\varphi(l)=\varphi_{xx}(0)=\varphi_{xx}(l)=0$.

Then by integrating by parts 3 times with respect to the variable x in the integral

$$\varphi_n = \left(\frac{2}{l}\right)^{1/2}\int_0^l \varphi(x)\sin\frac{\pi n}{l}x dx$$

we get

$$\varphi_n = -\left(\frac{l}{\pi}\right)^3\frac{p_n}{n^3}, \tag{31}$$

$$\sum_{n=1}^{\infty}p_n^2 \le \frac{4}{l^2}\int_0^l\left[\varphi_{xxx}(x)\right]^2 dx < \infty. \tag{32}$$

We note that formula (32) represents the Bessel inequality. Using (31) and (32), taking into account (29) and (30), now we can show that the series (27) and (28) converge absolutely and uniformly in the domain $\overline{\Omega}$. In this case termwise differentiation of these series (27) and (28) with respect to variables t and x is possible and the obtaining series will converge absolutely and uniformly in the domain $\overline{\Omega}$.

Indeed, using (29), (31) and (32) and applying the Minkowski and Hölder inequalities, for series (27) and (28) in the domain $\overline{\Omega}$ the following estimate is obtained:

$$\begin{aligned} \left|U(t,x)\right| &= \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left|u_n(t)\right|\left|\sin\mu_n x\right| \leq \sqrt{\frac{2}{l}\sum_{n=1}^{\infty}\left|C_{1n}\right|^2}\sqrt{\sum_{n=1}^{\infty}\left|\varphi_n\right|^2} \leq \\ &\leq \gamma_1\sum_{n=1}^{\infty}\frac{1}{n^3}\left|p_n\right| \leq \gamma_1\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^6}}\sqrt{\sum_{n=1}^{\infty}\left|p_n\right|^2} \leq \\ &\leq \frac{2\gamma_1}{l}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^6}}\sqrt{\int_0^l\left[\varphi_{xxx}(x)\right]^2dx} < \infty, \end{aligned} \tag{33}$$

where $\gamma_1 = \sqrt{\frac{2}{l}\sum_{n=1}^{\infty}\left|C_{1n}\right|^2}\left(\frac{l}{\pi}\right)^3$.

From (33) it follows that series (27) and (28) converge absolutely and uniformly in domain $\overline{\Omega}$. Formally differentiating functions (27) and (28) we obtain

$$U_{tt}(t,x,\nu) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\varphi_n\Phi_n''(t,\nu)\sin\mu_n x, \;\; t>0, \tag{34}$$

$$U_{tt}(t,x,\nu) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\varphi_n\Psi_n''(t,\nu)\sin\mu_n x, \;\; t<0, \tag{35}$$

$$U_{xx}(t,x,\nu) = -\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2\varphi_n\Phi_n(t,\nu)\sin\mu_n x, \;\; t>0, \tag{36}$$

$$U_{xx}(t,x,\nu) = -\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2\varphi_n\Psi_n(t,\nu)\sin\mu_n x, \;\; t<0, \tag{37}$$

$$U_{ttxx}(t,x,\nu) = -\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2\varphi_n\Phi_n''(t,\nu)\sin\mu_n x, \;\; t>0, \tag{38}$$

$$U_{ttxx}(t,x,\nu) = -\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\mu_n^2\varphi_n\Psi_n''(t,\nu)\sin\mu_n x, \;\; t<0, \tag{39}$$

where $\mu_n = \frac{\pi n}{l}$.

Analogously to (33), taking into account formulas (30)–(32) and applying the Minkowski and Hölder inequalities, for the series (34) and (35) in domain Ω we obtain the following estimate:

$$\begin{aligned} \left|U_{tt}(t,x)\right| &= \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\left|u_n''(t)\right|\left|\sin\mu_n x\right| \leq \sqrt{\frac{2}{l}\sum_{n=1}^{\infty}\left|C_{2n}\right|^2}\sqrt{\sum_{n=1}^{\infty}\left|\varphi_n\right|^2} \leq \\ &\leq \gamma_2\sum_{n=1}^{\infty}\frac{1}{n^3}\left|p_n\right| \leq \frac{2\gamma_2}{l}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^6}}\sqrt{\int_0^l\left[\varphi_{xxx}(x)\right]^2dx} < \infty, \end{aligned} \tag{40}$$

where $\gamma_2 = \sqrt{\frac{2}{l}\sum_{n=1}^{\infty}\left|C_{2n}\right|^2}\left(\frac{l}{\pi}\right)^3$.

Analogously, for the series (36) and (37) in the domain Ω we get the estimate

$$\begin{aligned}\left|U_{xx}(t,x)\right| = \frac{\sqrt{2}\pi^2}{l^2\sqrt{l}}\sum_{n=1}^{\infty} n^2\left|u_n(t)\right|\left|\sin\mu_n x\right| \leq \gamma_3\sqrt{\sum_{n=1}^{\infty} n^2\left|\varphi_n\right|^2} \leq \\ \leq \gamma_3\sum_{n=1}^{\infty}\frac{1}{n}\left|p_n\right| \leq \frac{2\gamma_3}{l}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\int_0^l\left[\varphi_{xxx}(x)\right]^2dx} < \infty,\end{aligned} \tag{41}$$

where $\gamma_3 = \sqrt{\frac{2}{l}\sum_{n=1}^{\infty}\left|C_{1n}\right|^2}\frac{\pi^2}{l^2}$.

Similarly, for the series (38) and (39), as in the estimates (40) and (41) in the domain Ω, we easily obtain that

$$\left|U_{ttxx}(t,x)\right| < \infty.$$

Consequently, the function $U(t,x)$ on Ω, defined by the series (27) and (28), satisfies the conditions of the problem.

To establish the uniqueness of the solution, we show that under the zero integral condition $\int_0^T U(t,x)dt = 0,\ \ 0 \leq x \leq l$, the boundary value problem (1)–(3) has only the trivial solution. We suppose that $\varphi(x) \equiv 0$. Then $\varphi_n = 0$ and from (27) and (28) in domain Ω we get

$$\int_0^l U(t,x)\sin\frac{\pi n}{l}xdx,\ n = 1,2,\ldots$$

Hence, by virtue of the completeness of systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin(\pi n/l)x\right\}$ in $L_2[0;\ l]$ we conclude, that $U(t,\ x) \equiv 0$ for all $x \in [0;\ l]$ and $t \in [-T;\ T]$.

Consequently, for regular values of the spectral parameter $\nu \in \Lambda$ the problem (1)–(3) has a unique solution in domain Ω.

Thus it is proved that the following theorem holds.

Theorem. Let the conditions A and B are satisfied. Then for regular values of the spectral parameter $\nu \in \Lambda$ the problem (1)–(3) is uniquely solvable in domain Ω.

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