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M a t h e m a t i c s

ON FIXED POINTS OF AUTOMORPHISMS

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We prove that each automorphism of order 2 of any non-abelian periodic group of odd period has a non-trivial fixed point.

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On Fixed Points of Automorphisms of Periodic Groups. In the Kourovka Notebook the following question was posed by V.D. Mazurov: "Let α be an automorphism of prime order q of an infinite free Burnside group $G = B(q, p)$ of prime exponent p such that α cyclically permutes the free generators of G. Is it true that α fixes some non-trivial element of G ?" (see [1]).

By definition free Burnside group $B(m,n)$ of period n and rank m has the following presentation:

$$B(m,n) = \langle b_1, b_2, ..., b_m \mid X^n = 1\rangle,$$

where X runs over the set of all words in the alphabet $\{b_1^{\pm 1}, b_2^{\pm 1}, \dots, b_m^{\pm 1}\}$. The group $B(m,n)$ is the quotient group of the free group F_m of rank m by the normal subgroup F_m^n generated by n powers of all elements from F_m (see [2]).

The automorphisms of these groups have less investigated. It was proved that all those normal automorphisms as well as all splitting automorphisms of order p^k (p is prime) are inner for all odd $n \geq 1003$ (see [3–6]). It is also known that for the same n the automorphism groups $\mathrm{Aut}(B(m,n))$ of $B(m,n)$ are complete, that is, has trivial center and all its automorphisms are inner (see [7]).

The positive answer to the Mazurov's question for all prime $p > 2$ and $q = 2$ follows from the following more general statement.

T h e o r e m . *Let ϕ be an arbitrary automorphism of order 2 (involution) of a periodic group G of odd period n. If $\phi(h) \neq h^{-1}$ for some $h \in G$, then $s = \phi(h)(h^{-1}\phi(h))^{\frac{n-1}{2}}$ is a non-trivial fixed point of ϕ.*

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Proof. First of all, we prove that the element $s=\phi(h)(h^{-1}\phi(h))^{\frac{n-1}{2}}$ is non-trivial. Assuming that $s=1$, we obtain the equality $(h^{-1}\phi(h))^{\frac{n-1}{2}}=\phi(h)^{-1}$ and so, $(h^{-1}\phi(h))^{n-1}=\phi(h)^{-2}$. By virtue of relation $(h^{-1}\phi(h))^n=1$ (recall that G is an n-periodic group) we get $\phi(h)=h^{-1}$. The obtained contradiction proves that $s\neq 1$.

Let us check that $\phi(s)=s$. We have

$$\phi(s)=\phi(\phi(h)(h^{-1}\phi(h))^{\frac{n-1}{2}})=\phi^2(h)(\phi(h)^{-1}\phi^2(h)^{\frac{n-1}{2}}. \tag{1}$$

Using the condition $\phi^2=1_G$ and the relation $(\phi(h)^{-1}h)^n=1$, from Equality (1) we get

$$\phi(s)=h(\phi(h)^{-1}h)^{\frac{n-1}{2}}=hh^{-1}\phi(h)(h^{-1}\phi(h))^{\frac{n-1}{2}}=$$
$$=h(h^{-1}\phi(h))^{\frac{n+1}{2}}=\phi(h)(h^{-1}\phi(h))^{\frac{n-1}{2}}=s.$$

Theorem is proved. □

Corollary 1. *Any automorphism of order 2 of a non-abelian periodic group G of odd period has a non-trivial fixed point.*

Proof. It is well known (and easy to prove), that is an automorphism $\phi: G\to G$ of a group G satisfies the condition $\phi(h)=h^{-1}$ for all $h\in G$, then G is an abelian group. □

Since for any $n\geq 3$ and $m>1$ the group $B(m,n)$ is non-abelian, we get a positive answer to the above mentioned question for all prime $p>2$ and $q=2$.

Corollary 2. *Any automorphism of order 2 of free Burnside group $B(m,n)$ has a non-trivial fixed point for all odd periods $n\geq 3$ and ranks $m>1$.*

We wood like to emphasize that Theorem allows us to construct fixed points of involutions. For example, if ϕ is the automorphism of $B(2,n)$ which permutes the free generators b_1 and b_2, then $s=b_1(b_1^{-1}b_2)^{\frac{n+1}{2}}$ is a fixed point for ϕ. To prove this, it is enough to choose $h=b_1$ in Theorem (it is obvious that $\phi(h)=b_2\neq h^{-1}=b_1^{-1}$). Choosing $h=b_1b_2$, we get another fixed point $s=b_1b_2[b_2,b_1]^{\frac{n+1}{2}}$ and so on.

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REFERENCES

1. Mazurov V.D., Khukhro E.I. *The Kourovka Notebook. Unsolved Problems in Group Theory.* Ross. Akad. Sci. Sib. Div. (17th ed.). Inst. Math. Novosibirsk (2010), 227 p.
2. Adian S.I. *The Burnside Problem and Identities in Groups*. Ergeb. Math. Grenzgeb. Springer-Verlag, Berlin–New York, **95** (1979), xi+311 p.
3. Cherepanov E.A. Normal Automorphisms of Free Burnside Groups of Large Odd Exponents. *Int. J. Algebra Comput.*, **16** : 5 (2006), 839–847.
4. Atabekyan V.S. Normal Automorphisms of Free Burnside Groups. *Izv. Math.*, **75** : 2 (2011), 223–237.

5. Atabekyan V.S. Splitting Automorphisms of Free Burnside Groups. *Sb. Math.*, **204** : 2 (2013), 182–189.
6. Atabekyan V.S. Splitting Automorphisms of Order p^k of Free Burnside Groups are Inner. *Math. Notes*, **95** : 5 (2014), 586–589.
7. Atabekyan V.S. The Groups of Automorphisms are Complete for Free Burnside Groups of Odd Exponents $n \geq 1003$. *Int. J. Algebra Comput.*, **23** : 6 (2013), 1485–1496.

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ԱՎՏՈՄՈՐՖԻԶՄՆԵՐԻ ԱՆՇԱՐԺ ԿԵՏԵՐԻ ՄԱՍԻՆ

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