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Mathematics

ON BLOWUP OF CERTAIN COVERING SPACES

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Using the finite-sheeted unbranched covering maps between the topological spaces we define the notion of the blowup of a topological space and show the existence of the blowups of certain covering spaces.

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1. Introduction. In the papers [1–3] the subspaces of the space $G \times [0,\infty)$, where G is some compact solenoidal group, have been considered and their covering spaces have been studied. Here we consider the product space $F \times D$, where F is some compact space and D is the closed unit disk in the complex plane $\mathbb{C}$. Then via the finite-sheeted covering maps we define the notion of a blowup of a topological space and show the existence of the blowups of covering spaces of $F \times D$.

Thus, let M, X and Y be topological spaces and let $\tau : Y \to M$ be an unbranched finite-sheeted covering. Also let $\pi : X \to M$ be a covering map with a set of "critical" points $K \subset M$, so that π is an unbranched finite-sheeted covering over $M \setminus K$ (see [4], p. 25−26).

Definition 1. The space Y will be called a blowup of the space X, if there exists a mapping $\varphi : Y \to X$ such that the restriction of φ to $Y^* = \tau^{-1}(M^*)$, where $M^* = M \setminus K$ is a homeomorphism between the spaces Y^* and $X^* = \pi^{-1}(M^*)$ and the diagram

$$\begin{array}{ccc} X & \xleftarrow{\;\varphi\;} & Y \\ & {}_{\pi}\searrow \quad \swarrow_{\tau} & \\ & M & \end{array}$$

commutes.

Let D denote the closed unit disk in the complex plane $\mathbb{C}$ and let S denote the unit circle in $\mathbb{C}$. Suppose F is some compact space. Consider a cylinder $M = F \times D$

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with "lateral side" $bM = F \times S$. A set $K \subset M$ that is closed in M will be called a *thin* set, if:

1. $bM \cap K = \varnothing$ and
2. for each fixed $x \in F$ the set $M_x = \{z \in D : (x,z) \in K\}$ is finite.

Let $\pi : L \to M$ be a covering such that for some thin set K the restriction $\pi^* : L^* \to M^*$ of π on $L^* = \pi^{-1}(M^*)$, where $M^* = M \setminus K$, is an n-fold unbranched covering. For each $x \in F$ denote

$$D_x = \{x\} \times D, \quad D_x^* = D_x \setminus K$$

and

$$S_x = \{x\} \times S, \quad V_x = \pi^{-1}(S_x).$$

Let $\gamma : I = [0,1] \to M$ be a continuous mapping, which determines the continuous path $\gamma(I)$ in M.

Definition 2. A path $\gamma(I)$ in M will be called analytic, if it is entirely contained in some set $D_x^*, x \in F$.

As $S_x \subset D_x^*$, then each path that is entirely contained in S_x, $x \in F$, is analytic.

By path lifting property (see [5], p. 282) for each path $\gamma(I)$ in M^* and for each point $w \in \pi^{-1}(\gamma(0))$ there exists a unique path $\hat{\gamma}(I) \subset L^*$, which starts at w and lies over the path $\gamma(I)$, i.e., $\hat{\gamma}(0) = w$ and $\gamma(t) = \pi \circ \hat{\gamma}(t)$, $t \in I$. The path $\hat{\gamma}(I)$ is called the lift of $\gamma(I)$.

Definition 3. A path in L^* will be called analytic, if it is a lift of some analytic path from M^*.

In particular, any path in L that is entirely contained in some set $V_x = \pi^{-1}(S_x)$ is analytic.

Let us now introduce the notion of equivalent points in the space L^*.

Definition 4. Two points $w_1, w_2 \in L^*$ will be called equivalent, if $\pi(w_1) = \pi(w_2)$ and there exists an analytic path $\hat{\gamma}(I) \subset L^*$ such that $w_1 = \hat{\gamma}(0)$ and $w_2 = \hat{\gamma}(1)$.

The equivalence of points w_1 and w_2 is denoted by $w_1 \sim w_2$. It is easy to check that for any points w_1, w_2 and w_3 from L^* we have that if $w_1 \sim w_2$ and $w_2 \sim w_3$, then $w_1 \sim w_3$. The properties of reflexivity and symmetry of the presented relation are obvious. Thus, for each $(x,z) \in M^*$ the set $\pi^{-1}(x,z) = \{w_1, ..., w_n\}$ breaks up into the finite number of equivalence classes. On L^* define a function $\nu : L^* \to \mathbb{Z}_+$ as follows:

$$\nu(w) = \text{card}\{w' \in \pi^{-1}(\pi(w)) : w' \sim w\}, \quad w \in L^*.$$

Thus, the function $\nu : L^* \to \mathbb{Z}_+$ assigns to each point from L^* the number of its equivalent points. Since $w \sim w$, then, clearly, $\nu(w) \geq 1, w \in L^*$.

Let $\pi_1 : bL \to bM$ be a restriction of the covering $\pi : L \to M$ to $bL = \pi^{-1}(bM)$. Since $bM \subset M^*$, then π_1 is an unbranched n-fold covering.

Finally, let us define the set $E = \pi^{-1}(F \times \{1\})$ and an n-fold unbranched covering $\pi_2 : E \times S \to bM$, setting

$$\pi_2((y,\xi)) = (x,\xi)$$

for $(y,\xi)\in E\times S$ with $\pi(y)=(x,1)$.

Lemma. Suppose that $\nu(w)=1$ for any $w\in bL$. Then there exists a homeomorphism $\sigma: bL\to E\times S$ such that the diagram

$$\begin{array}{ccc} bL & \xrightarrow{\ \sigma\ } & E\times S \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & bM & \end{array}$$

commutes.

Proof. Define a mapping $\sigma: bL\to E\times S$ as follows. Let $w\in bL$ and assume $\pi_1(w)=(x,\xi)$, $x\in F$, $\xi\in S$. Consider a continuous mapping $\gamma: I\to\{x\}\times S=S_x$, which determines a path $\gamma(I)$ with $\gamma(0)=(x,1)$ and $\gamma(1)=(x,\xi)$. Let $\hat{\gamma}: I\to bL$ be the lift of γ in bL: $\gamma=\pi_1\circ\hat{\gamma}$, with $\hat{\gamma}(1)=w$. Set $y:=\hat{\gamma}(0)\in E$ and define $\sigma(w):=(y,\xi)\in E\times S$. In other words, given a point $w\in bL$ we consider the image $(x,\xi)=\pi_1(w)$ and construct in S_x the path γ connecting $(x,1)$ with (x,ξ).

Then we take the lift $\hat{\gamma}$, which passes through w. Denote its initial point by y and define $\sigma(w)=(y,\xi)$. Let us show that σ is well-defined. First, since $\pi_1: bL\to bM$ is an unbranched n-fold covering, there is a unique lift $\hat{\gamma}$ of γ passing through w. Further, assume that $\gamma_1(I)$ is another continuous path in S_x connecting $(x,1)$ with (x,ξ). Since the paths $\gamma(I)$ and $\gamma_1(I)$ are entirely contained in S_x, they are both analytic.

Therefore, their lifts $\hat{\gamma}$ and $\hat{\gamma}_1$ are analytic as well, i.e., if $\hat{\gamma}_1(1)=w$, then $\hat{\gamma}_1(0)=\hat{\gamma}(0)$. Indeed, since $\pi_1\circ\hat{\gamma}(0)=\gamma(0)=\pi_1\circ\hat{\gamma}_1(0)$, the assumption $\hat{\gamma}_1(0)\neq\hat{\gamma}(0)$ would mean that distinct points $\hat{\gamma}_1(0)$ and $\hat{\gamma}(0)$ are equivalent since they are the endpoints of an analytic path which connects $\hat{\gamma}_1(0)$ with w and then w with $\hat{\gamma}(0)$. But this contradicts the condition $\nu(w)=1, w\in bL$. Thus, the mapping σ is well-defined. By construction we have that $\pi_1=\pi_2\circ\sigma$. Let us show that σ is bijective. Suppose $\sigma(w_1)=\sigma(w_2):=(y,\xi)\in E\times S$. Let us say that $\pi_1(y)=(x,1)$ with $x\in F$. Then $\pi_1(w_1)=\pi_1(w_2)=(x,\xi)$ and by the construction of σ we have that w_1 and w_2 are the endpoints of some analytic paths, which start at y. Since $\nu(w_1)=1$, we get $w_1=w_2$. Hence, σ is injective. Now suppose that $(y,\xi)\in E\times S$ and $\pi_1(y)=(x,1)$, $x\in F$. Consider any path $\gamma: I\to S_x$ with $\gamma(0)=(x,1)$ and $\gamma(1)=(x,\xi)$. Since $y\in\pi_1^{-1}(\gamma(0))$, by path lifting property there exists a unique lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0)=y$. Then $\pi_1\circ\hat{\gamma}(1)=\gamma(1)=(x,\xi)$. Denoting $w=\hat{\gamma}(1)$, we get that $\sigma(w)=(y,\xi)$, i.e. σ is surjective and therefore bijective. Continuity and openness of the mapping σ follow from the continuity and the openness of the coverings π_1 and π_2 and from their local homeomorphity (see the proof of Theorem 3.1 from [2]). $\square$

The Theorem is proved similarly.

Theorem. Suppose $\nu(w)=1$ for any $w\in L^*$. Then there exists a blowup $E\times D$ of the space L such that the diagram

$$\begin{array}{ccc} L & \xleftarrow{\ \varphi\ } & E\times D \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & M & \end{array}$$

commutes, where $\pi_1 : E \times D \to M : (y,z) \mapsto (x,z)$ with $(x,1) = \pi(y)$ is an unbranched n-fold covering and $\varphi : E \times D \to L$ is the blowup mapping.

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REFERENCES

1. **Grigorian S.A., Gumerov R.N., Kazantsev A.V.** Group Structure in Finite Coverings of Compact Solenoidal Groups. // Lobachevskii Journal of Mathematics 2000, v. 6, p. 39–46.
2. **Beknazaryan A.F., Grigoryan S.A.** On Bohr-Riemann Surfaces. // Proceedings of the NAS Armenia, 2014, v. 49, № 5, p. 76–88 (in Russian).
3. **Beknazaryan A.F.** Topologies on the Generalized Plane. // Proceedings of the Yerevan State University. Physical and Mathematical Sciences, 2014, № 3, p. 8–12.
4. **Forster O.** Riemann Surfaces. Russian translation. M.: Mir, 1980.
5. **Lee J.M.** Introduction to Topological Manifolds (2nd ed.). Graduate Texts in Mathematics. NY: Springer, 2011, v. 202.