PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2013, № 3, p. 29–36

Mathematics

DUALITY IN SOME SPACES OF FUNCTIONS HARMONIC IN THE UNIT BALL

A. I. PETROSYAN *, E. S. MKRTCHYAN

Chair of the General Mathematics YSU, Armenia

We introduce the Banach spaces $h_\infty(\varphi)$, $h_0(\varphi)$ and $h^1(\eta)$ of functions harmonic in the unit ball in $\mathbb{R}^n$, depending on weight function φ and weighting measure η. The paper studies the following question: for which φ and η we have $h^1(\eta)^* \sim h_\infty(\eta)$ and $h_0(\varphi)^* \sim h^1(\eta)$. We prove that the necessary and sufficient condition for this is that certain linear operator, which projects $L^\infty(d\eta\, d\sigma)$ onto the subspace $\varphi h_\infty(\varphi)$, is bounded.

MSC2010: 30H05; 46E15.

***Keywords*:** Banach space, harmonic function, weight function, weighting measure, bounded projector.

Introduction. A positive continuous decreasing function φ on $[0,1)$ is called a weight function, if

$$\lim_{r\to 1}\varphi(r)=0.$$

Positive finite Borel measure η on $[0,1)$ is called a weighting measure, if it is not supported in any subinterval $[0,\rho)$, $0<\rho<1$. Let $h_\infty(\varphi)$ be the Banach space of complex-valued functions u harmonic in the unit disc with the norm $\|u\|_\varphi = \sup\{|u(z)|\varphi(|z|):\ |z|<1\}$ and let $h_0(\varphi)$ be its closed subspace of functions u with $|u(z)| = o\big(1/\varphi(|z|)\big)$, as $|z|\to 1$.

It has been shown by Rubel and Shields [1], that $h_\infty(\varphi)$ is isometrically isomorphic to the second dual of $h_0(\varphi)$. The duality problem of finding a weighting measure η such that

$$h^1(\eta)=\big\{v\in L^1(d\eta(r)\,d\sigma):\ v \text{ is harmonic in } |z|<1\big\}$$

represents the intermediate space, the dual of $h_0(\varphi)$ and the predual of $h_\infty(\varphi)$, i.e. $h^1(\eta)\sim h_0(\varphi)^*$ and $h^1(\eta)^*\sim h_\infty(\eta)$ was posed and solved in [2]. The mentioned articles [1] and [2] tread only the case $n=2$. It is well-known that in this case every harmonic function h has an expansion to a series in terms of z and $\bar z$ in the unit disk $|z|<1$, since every real-valued harmonic function is the real part of some holomorphic function. This allows to apply the methods of complex analysis.

* E-mail: apetrosyan@ysu.am

In the present work we consider the duality problem in the case of harmonic functions in the unit ball of $\mathbb{R}^n$, $n>2$. In the multidimensional case we can not to speak about connection between harmonic and holomorphic functions, and instead of expansion in z and $\bar{z}$ we deal with expansions in spherical harmonics.

Below the following notation is used: $B=\{x\in\mathbb{R}^n:\ |x|<1\}$ is the open unit ball in $\mathbb{R}^n$; $S=\{x\in\mathbb{R}^n:\ |x|=1\}$ is the unit sphere in $\mathbb{R}^n$; $h(B)$ is the vector space of complex-valued functions harmonic in B with the usual pointwise addition and scalar multiplication; σ stands for the Lebesgue measure of the area element on S normed by the condition $\sigma(S)=1$.

Spaces of Harmonic Functions. Let $\varphi(r)$ be a weight function and η be an weighting measure. We extend φ to the whole B by $\varphi(x)=\varphi(|x|)$. For $u\in h(B)$ we define

$$\|u\|_\varphi=\sup\big\{|u(x)|\varphi(x):\ x\in B\big\}=\sup\big\{M_\infty(u,r)\varphi(r):\ r<1\big\},$$

$$\|u\|_\eta=\int\limits_S\int\limits_0^1|u(r\zeta)|\,d\eta(r)d\sigma(\zeta)=\int\limits_0^1 M_1(u,r)\,d\eta(r),$$

where $M_\infty(u,r)=\sup\{|u(x)|:\ |x|=r\}$, $M_1(u,r)=\int\limits_S|u(r\zeta)|\,d\sigma(\zeta)$. We define the following spaces of harmonic functions:

$h_\infty(\varphi)=\big\{u\in h(B):\ \|u\|_\varphi<\infty\big\}$,
$h_0(\varphi)=\big\{u\in h(B):\ \lim_{r\to1}M_\infty(u,r)\varphi(r)=0\big\}$,
$h^1(\eta)=\big\{u\in h(B):\ \|u\|_\eta<\infty\big\}$.

Obviously, $h_0(\varphi)\subset h_\infty(\varphi)$, so, we can use the norm $\|u\|_\varphi$ on $h_0(\varphi)$.

In the following two propositions we establish some basic facts about these spaces.

Proposition 1. Let h denote any of the spaces $h_\infty(\varphi)$, $h_0(\varphi)$ or $h^1(\eta)$. Then:

(i) if b is a bounded subset of h, then the functions in b are uniformly bounded on each compact subset of B;

(ii) if the sequence u_n converges in h, then it converges uniformly on each compact subset of B;

(iii) for any point in B, the value at that point is a bounded linear functional on h;

(iv) h is a Banach space;

(v) $h_0(\varphi)$ is a closed subspace of $h_\infty(\varphi)$.

Proof. In [3] (Proposition 2) in terms of our notations, the following inequality is received for $u\in h^1(\eta)$:

$$|u(x)|\le\frac{2^n}{(1-|x|)^{(n-1)}}\left(\int\limits_{(1+|x|)/2}^1|d\eta(t)|\right)^{-1}\|u\|_\eta,\qquad x\in B.$$

This gives us (i) and (iii) for $u\in h^1(\eta)$. For $h_\infty(\varphi)$ and $h_0(\varphi)$ these statements are obvious. It is easy to see also that (ii) follows from (i).

Now we prove (iv). The spaces h are obviously linear and normed, therefore, it is enough to prove their completeness. First we prove the completeness of $h^1(\eta)$. Let the sequence u_j be fundamental in $h^1(\eta)$, and let K be a compact subset of B. From (ii) we deduce that there is a constant $C = C(K)$ such that $\max_{x\in K}|u(x)| \le C\|u\|_\eta$ for all $u \in h^1(\eta)$. Therefore, $|u_j(x) - u_k(x)| \leqslant C\|u_j - u_k\|_\eta$ for any $x \in K$ and j, k. As u_j is fundamental in $h^1(\eta)$, then the sequence u_j converges uniformly to some harmonic in B function u on compact subsets of B. Again, by the fundamentality of u_j, we have

$$\int_S \int_0^1 |u_j(r\zeta)|\, d\eta(r) d\sigma(\zeta) = \|u_j\|_\eta \le \|u_j - u_k\|_\eta + \|u_k\|_\eta \le C.$$

By Fatou's lemma, $\|u\|_\eta = \int_S \int_0^1 |u(r\zeta)|\, d\eta(r) d\sigma(\zeta) \le C,$ i.e. $u \in h^1(\eta)$.

The case of $h_\infty(\varphi)$ is easier. Let $u_j \in h_\infty(\varphi)$. The function $\varphi(x)$ is away from zero on a compact subset K, hence,

$$|u_j(x) - u_k(x)| \le C\varphi(x)|u_j(x) - u_k(x)| = C\|u_j - u_k\|_\varphi, \quad x \in K.$$

As the sequence u_j is fundamental in $h_\infty(\varphi)$, u_j converges uniformly to some function u on compact subsets of B, which is harmonic in B. Obviously, u_j converges to u also in $h_\infty(\varphi)$.

Part (v) follows from (iv). □

Proposition 2. Let $u_\rho(x) = u(\rho x)$, $0 \le \rho \le 1$.

(i) If $u \in h^1(\eta)$ or $u \in h_0(\varphi)$, then $u_\rho \to u$ in norm as $\rho \to 1$;

(ii) if $u \in h_\infty(\varphi)$, then $\|u_\rho\|_\varphi \le \|u\|_\varphi$ and $u_\rho \to u$ pointwise in B;

(iii) the harmonic polynomials are dense in $h^1(\eta)$ and in $h_0(\varphi)$;

(iv) each $u \in h_\infty(\varphi)$ is pointwise limit of some sequence of polynomials bounded in norm.

Proof. (i). The statement is obvious for $h_\infty(\varphi)$. Consider the case of $h^1(\eta)$. For any $\delta \in (0, 1)$ we have

$$|u_\rho - u|_\eta = \int_0^\delta \int_S |u(\rho r\zeta) - u(r\zeta)|\, d\sigma(\zeta)\, d\eta(r) \le \tag{1}$$

$$\le \int_0^\delta \int_S |u(\rho r\zeta) - u(r\zeta)|\, d\sigma(\zeta)\, d\eta(r) + \int_\delta^1 \int_S \big(|u(\rho r\zeta)| + |u(r\zeta)|\big)\, d\sigma(\zeta)\, d\eta(r).$$

Due to subharmonicity of $|u|$, the mean value along the unit sphere

$$m(\rho) = \int_S |u(\rho r\zeta)|^p d\sigma(\zeta)$$

is nondecreasing function of ρ: $m(\rho) \leqslant m(1)$. Hence, using (1), we obtain

$$\|u_\rho - u\|_\eta \le \int_0^\delta \int_S |u(\rho r\zeta) - u(r\zeta)|\, d\sigma(\zeta)\, d\eta(r) + 2\int_\delta^1 \int_S |u(r\zeta)|\, d\sigma(\zeta) d\eta(r),$$

so, first by choosing the number δ, and then ρ close to 1, the right hand side of this inequality can be made arbitrary small.

(ii) is immediate from the maximum principle and continuity.

As it is well known, any function harmonic in a neighbourhood of $\overline{B}$ can be approximated uniformly on $\overline{B}$ by harmonic polynomials. Using this fact, we deduce (iii) and (iv). □

We note also that in the [4] the appropriate properties of spaces of holomorphic functions in the ball of $\mathbb{C}^n$ are given.

The Reproducing Kernel. First consider the case $n = 2$. If a function u is harmonic in the unit disc, then it is a real part of holomorphic function f. As $u = (f + \overline{f})/2$, we obtain from the expansion in power series

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}, \tag{2}$$

where $0 \le r < 1$. In the multidimensional case (i.e., if $n > 2$) there is no such connection between harmonic and holomorphic functions, but there is an analog of expansion (2), in which the role of exponentials $e^{ik\theta}$ play the spherical harmonics. Now we recall some facts from the theory of harmonic functions (for example, [5]) concerning this expansion.

Let $\mathcal{H}_k(\mathbb{R}^n)$ denote the space of all complex-valued homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree k.

A *spherical harmonic* of degree k is the restriction to S of an element of $\mathcal{H}_k(\mathbb{R}^n)$. The collection of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k(S)$.

Any spherical harmonic $p \in \mathcal{H}_k(S)$ has a unique extension to an element of $\mathcal{H}_k(\mathbb{R}^n)$, we will denote this extension by p as well.

$\mathcal{H}_k(S)$ is a finite-dimensional closed subspace of Hilbert space $L^2(S)$ with inner product $(u,v) = \int\limits_S u(\zeta)\overline{v(\zeta)}\, d\sigma(\zeta)$.

Fix a point $\xi \in S$ and consider the map $\Lambda\colon \mathcal{H}_k(S) \to \mathbb{C}$, defined by $\Lambda(p) = p(\xi)$. Λ is clearly a linear functional. By the self-duality of the finite-dimensional Hilbert space $\mathcal{H}_k(S)$, there exist a unique $Z_k(\cdot,\xi) \in \mathcal{H}_k(S)$ such that

$$p(\xi) = \int\limits_S p(\zeta) Z_k(\zeta,\xi)\, d\sigma(\zeta)$$

for all $p \in \mathcal{H}_k(S)$. The function Z_k is called the zonal harmonic of degree k with pole ξ. It is real valued and satisfies $Z_k(\zeta,\xi) = Z_k(\xi,\zeta)$.

Let φ be any weight function and η any weighting measure. We introduce the measure $d\mu = \varphi\, d\eta$ and the measure $d\mu'$ defined by $\int\limits_0^1 f(r)\, d\mu'(r) = \int\limits_0^1 f(r^2)\, d\mu(r)$, $f \in C[0,1]$. Clearly, both μ and μ' are finite positive Borel measures on $[0,1)$ and are not supported on any subinterval $[0,\rho)$, $0 < \rho < 1$.

Let $x, y \in \mathbb{R}^n$ and $x = r\zeta,\ y = \rho\xi$ be their polar form, where $r = |x|$, $\rho = |y|$, $\zeta,\ \xi \in S$. We introduce the function

$$R_\mu(x,y) = \sum_{k=0}^{\infty} t_k^{-1} r^k \rho^k Z_k(\zeta,\xi) = \sum_{k=0}^{\infty} t_k^{-1} Z_k(x,y), \tag{3}$$

where $t_k = \int_0^1 r^k\, d\mu'(r) = \int_0^1 r^{2k}\, d\mu(r)$. In the second equality of (3), $Z_k(x,y)$ is the extension of $Z_k(\zeta,\xi)$ in each variable from S to an element of $\mathcal{H}_k(\mathbb{R}^n)$, therefore, $r^k\rho^k Z_k(\zeta,\xi) = Z_k(x,y)$.

We shall use the duality relation between $h_\infty(\varphi)$ and $h^1(\eta)$, which is given by the bilinear form

$$\langle u,v\rangle = \int_S \int_0^1 u(r\zeta)\overline{v(r\zeta)}\varphi(r)\, d\eta(r)\, d\sigma(\zeta), \quad u \in h_\infty(\varphi),\ v \in h^1(\eta). \tag{4}$$

The following Proposition establishes that $\langle u(\cdot),\, R_\mu(x,\cdot)\rangle = u(x)$ for all $u \in h_\infty(\varphi) \cup h^1(\eta)$. This means that the function $R_\mu(x,y)$ is the reproducing kernel associated with the bilinear form (4).

Proposition 3.

a) For fixed $x \in B$ the function $R_\mu(x,y)$ is harmonic in the ball $\{y\colon\ |y| < |x|^{-1}\}$.

b) $R_\mu(x,y)$ is the reproducing kernel associated with the bilinear form (4), i.e. for all $u \in h_\infty(\varphi) \cup h^1(\eta)$ we have

$$\langle u(\cdot), R_\mu(x,\cdot)\rangle = u(x). \tag{5}$$

Proof. We have

$$t_k^{-1} r^k \rho^k |Z_k(\zeta,\xi)| \leqslant t_k^{-1} r^k \rho^k d_k, \tag{6}$$

where d_k is the dimension of the space $\mathcal{H}_k(S)$. As it is known [5],

$$d_k \leqslant C k^{n-2}, \tag{7}$$

where C is an absolute constant. As the measure μ' does not vanish in any neighbourhood of 1, then

$$t_k \geq \int_s^1 r^k d\mu'(r) \geq s^k \int_s^1 d\mu'(r), \quad k = 0,1,2,\ldots,$$

implying that $t_k^{-1} = O(s^{-k})$, $k \to \infty$ for each $0 < s < 1$. Hence, using (6) and (7), we obtain that the series in the right hand side of (3) converges absolutely and uniformly on the set $\{(x,y) \in \mathbb{R}^{2n}\colon\ |x||y| \leqslant q,\ 0 < q < 1\}$, implying the statement a).

Taking into account the equality $\varphi\, d\eta = d\mu$, (4) gives us that (5) is an integral representation: for any function $u \in h_\infty(\varphi)$ or $u \in h^1(\eta)$,

$$u(x) = \int_S \int_0^1 u(r\zeta) R_\mu(x, r\zeta)\, d\mu(r)\, d\sigma(\zeta). \tag{8}$$

We have $\|u\|_\mu = \int\limits_S \int\limits_0^1 |u(r\zeta)|\, d\mu(r) d\sigma(\zeta) = \int\limits_S \int\limits_0^1 |u(r\zeta)| \varphi(r)\, d\eta(r) d\sigma(\zeta)$, hence,

$$\|u\|_\mu \le \|u\|_\varphi \int\limits_0^1 d\eta(r) \le C\|u\|_\varphi, \tag{9}$$

$$\|u\|_\mu \le \max \varphi(r) \int\limits_S \int\limits_0^1 |u(r\zeta)|\, d\eta(r)\, d\sigma(\zeta) \le C\|u\|_\eta. \tag{10}$$

From (9) and (10) we deduce, that if $u \in h_\infty(\varphi) \cup h^1(\eta)$, then $u \in h^1(\mu)$. For functions from $h^1(\mu)$ (in a slightly different notations) the formula (8) is proved in [4] (Theorem 1), so, we get statement b). $\square$

Proposition 4. Let $\mathfrak{L}(R_\mu(x,\cdot))$ be the linear subspace spanned by the functions $\{R_\mu(x,\cdot), x \in B\}$. Then $\mathfrak{L}(R_\mu(x,\cdot))$ is dense in $h^1(\eta)$ and in $h_0(\varphi)$.

Proof. We will prove only for the case of $h^1(\eta)$, since the proof is similar for $h_0(\varphi)$. By the Hahn-Banach theorem it suffices to show, that if $\Phi \in h^1(\eta)^*$ and $\Phi(R_\mu(x,\cdot)) = 0$ for all $x \in B$, then Φ vanishes on $h^1(\eta)$.

As $R_\mu(x,y)$ is harmonic in $|y| < |x|^{-1}$, the series (3) converges in $\overline{B}$ uniformly and consequently, also by the norm of $h^1(\eta)$. Therefore,

$$\Phi\big(R_\mu(x,\cdot)\big) = \sum_{k=0}^{\infty} t_k^{-1} \Phi\big(Z_k(x,\cdot)\big) \tag{11}$$

for any $x \in B$. Let $\left\{e_1^{(k)}, \dots, e_{d_k}^{(k)}\right\}$ be an orthonormal bases in $\mathcal{H}_k(S)$. Then $Z_k(x,y) = \sum_{j=1}^{d_k} \bar{e}_j^{(k)}(x) e_j^{(k)}(y)$. We have

$$\Phi\big(Z_k(x,\cdot)\big) = \sum_{j=1}^{d_k} c_j^{(k)} \bar{e}_j^{(k)}(x), \tag{12}$$

where $c_j^{(k)} = \Phi(e_j^{(k)})$. Thus, $\Phi\big(Z_k(x,\cdot)\big) \in \mathcal{H}_k(\mathbb{R}^n)$ and (11) is the decomposition of function $\Phi\big(R_\mu(x,\cdot)\big)$ by homogeneous polynomials. We need to show that the equality $\Phi\big(R_\mu(x,\cdot)\big) = 0$ implies $\Phi\big(Z_k(x,\cdot)\big) = 0$, $k = 0,1,\dots$ Fix $\zeta \in S$ and consider the restriction of the function $\Phi\big(R_\mu(x,\cdot)\big)$ on a line segment $x = r\zeta$, $0 \le r < 1$.

We have $\Phi\big(R_\mu(r\zeta,\cdot)\big) = \sum_{k=0}^{\infty} t_k^{-1} \Phi\big(Z_k(r\zeta,\cdot)\big) = \sum_{k=0}^{\infty} t_k^{-1} \Phi\big(Z_k(\zeta,\cdot)\big) r^k \equiv 0$ for all $0 \le r < 1$.

By uniqueness theorem for coefficients of one variable power series $\Phi\big(Z_k(\zeta,\cdot)\big) = 0$ (as $t_k^{-1} \ne 0$), $k = 0,1,\dots$ Hence, by (12), we get $\Phi\big(Z_k(\zeta,\cdot)\big) = \sum_{j=1}^{d_k} c_j^{(k)} \bar{e}_j^{(k)}(\zeta) = 0$, $\zeta \in S$, $k = 0,1,\dots$, and using the linear independence of the system $\left\{\bar{e}_j^{(k)}\right\}_{j=1}^{d_k}$, we obtain that $c_j^{(k)} = 0$, $j = 1,\dots,d_k$, for all $k = 0,1,\dots$ Thus, Φ annihilates all harmonic polynomials, as any harmonic polynomial is a finite linear combination of $e_j^{(k)}$. According to part (iii) of Proposition 2, Φ vanishes on $h^1(\eta)$. $\square$

For a weighting measure η let $L^1(d\eta\, d\sigma)$ and $L^\infty(d\eta\, d\sigma)$ denote, respectively, the Banach spaces of complex-valued integrable and essentially bounded measurable functions with respect to the measure $d\eta\, d\sigma$ on B. Denote the norms of these spaces by $\|\cdot\|_\eta$ and $\|\cdot\|_\infty$ respectively. Let $C_0(B)$ be the Banach space of complex-valued continuous functions on $\overline{B}$ that vanish on S with the supremum norm. It is well known that the dual of $C_0(B)$ is $M(B)$, the space of finite complex Borel measures on B with the total variation norm. We identify $L^1(d\eta\, d\sigma)$ with the absolutely continuous measures with respect to $d\eta\, d\sigma$.

The main result of the present paper is the following theorem.

Theorem. Let φ be a weight function, η be a weighting measure and R_μ be the corresponding reproducing kernel (3). Consider the integral linear operators

$$(Tf)(x) = \int_S \int_0^1 R_\mu(x, r\zeta) f(r\zeta)\, d\eta(r)\, d\sigma(\zeta), \quad f \in L^\infty(d\eta\, d\sigma),$$

$$(S\nu)(x) = \int_B R_\mu(x,y)\varphi(y)\, d\nu(y), \quad \nu \in M(B).$$

The following conditions are equivalent:

(i) $\|R_\mu(x,\cdot)\|_\eta \le \dfrac{c}{\varphi(x)}, \quad x \in B$;

(ii) T is a bounded operator from $L^\infty(d\eta\, d\sigma)$ into $h_\infty(\varphi)$;

(iii) S is a bounded operator from $M(B)$ into $h^1(\eta)$;

(iv) $h^1(\eta)^* \sim h_\infty(\varphi)$;

(v) $h_0(\varphi)^* \sim h^1(\eta)$.

Proof. The implication (i) $\Rightarrow$ (ii) is immediate from the definition of T. Indeed,

$$|(Tf)(x)| \le \|f\|_\infty \int_S \int_0^1 |R_\mu(x, r\zeta)|\, d\eta(r)\, d\sigma(\zeta) = \|f\|_\infty \|R_\mu(x,\cdot)\|_\eta \le \|f\|_\infty \frac{c}{\varphi(x)},$$

implying $\|Tf\|_\varphi \le c\|f\|_\infty$.

It follows from Fubini's theorem, that (i) implies (iii). To prove this, note that

$$|S\nu|_\eta = \int_B \left| \int_B R_\mu(r\zeta, y)\varphi(y)\, d\nu(y) \right| d\eta(r)\, d\sigma(\zeta) \le$$

$$\le \int_B \left(\int_B |R_\mu(r\zeta, y)| d\eta(r)\, d\sigma(\zeta) \right) \varphi(y)\, d|\nu|(y) \le \int_B \frac{c}{\varphi(y)} \varphi(y)\, d|\nu|(y) = c\,|\nu|.$$

We now show that (iii) implies (v). The condition (v) means that each element from $h^1(\eta)$ can be identified with some linear functional on $h_0(\varphi)$ by means of a duality relation (4). More exactly, if we define a functional $l_v(u) = \langle u, v\rangle$, $u \in h_0(\varphi)$ for a given function $v \in h^1(\eta)$, then $l_v \in h_0(\varphi)^*$. Conversely, for each functional $l \in h_0(\varphi)^*$ there exist a unique $v \in h^1(\eta)$ such that $l = l_v$. Besides, the norms $\|l_v\|$ and $\|v\|_\eta$ are equivalent.

Indeed, according to the Holder's inequality, $|l_v(u)| = |\langle u, v\rangle| \le \|v\|_\eta\ \|u\|_\varphi$.

Thus, $l_v \in h_0(\varphi)^*$ and $\|l_v\| \le \|v\|_\eta$. It remains to prove that there exist a constant c such that each functional $l \in h_0(\varphi)^*$ is represented as $l = l_v$, where $v \in h^1(\eta)$ and $\|v\|_\eta \le c\,\|l\|$.

Regarding $h_0(\varphi)$ as a subspace of $C_0(B)$ by identifying $u \in h_0(\varphi)$ with $u\varphi \in C_0(B)$, we can use the Hahn-Banach theorem to prove that there exist $\nu \in M(B)$ such that $\|l\| = \|\nu\|$ and $l(u) = \int\limits_B u(y)\varphi(y)\,d\nu(y)$. Thus,

$$l(R_\mu(x,\cdot)) = \int\limits_B R_\mu(x,y)\varphi(y)\,d\nu(y) = (S\nu)(x).$$

Let $v = S\nu$. By (iii), $v \in h^1(\eta)$ and $\|v\| \leq \|S\|\,\|\nu\| = \|S\|\,\|l\|$. Also, by Proposition 3, $l_v(R_\mu(x,\cdot)) = \langle R_\mu(x,\cdot), v\rangle = v(x) = (S\nu)(x) = l(\,R_\mu(\,x,\,\cdot))$. Thus, the functionals l and l_v agree on the linear subspace $\mathfrak{L}(\,R_\mu\,(\,x,\,\cdot))$ spanned by the functions $\{R_\mu(x,\cdot), x \in B\}$. By Proposition 4, $l = l_v$. The uniqueness also follows from Proposition 4, since v is determined by the reproducing kernel.

The proof that (ii) implies (iv) is similar, using the duality of $L^1(d\eta\,d\sigma)$ and $L^\infty(d\eta\,d\sigma)$ instead of $C_0(B)$ and $M(B)$.

Assuming (iv), the Hahn-Banach theorem gives

$$\|R_\mu(x,\cdot)\|_\eta \leq c\,\sup\{|\langle u, R_\mu(x,\cdot)\rangle| : \ u \in h_\infty(\varphi), \|u\|_\varphi \leq 1\} =$$
$$= c\,\sup\{|u(x)| : \ u \in h_\infty(\varphi), \|u\|_\varphi \leq 1\} \leq \frac{c}{\varphi(x)}, \quad x \in B.$$

Similarly, (v) implies (i). □

It follows immediately from the reproducing property of R_μ that, when bounded, S is a projection from $M(B)$ onto the subspace $h^1(\eta)$.

Now consider the operator T. If $u \in h_\infty(\varphi)$, then $T(u\varphi) = u$. This follows from the reproducing property of R_μ. Suppose T is bounded. Then $\|u\|_\varphi \leq \|T\|\,\|u\varphi\|_\infty$. Consequently, $\varphi h_\infty(\varphi)$ can be regarded as a closed subspace of $L^\infty(d\eta\,d\sigma)$. Thus, when T is bounded, we can regard the operator φT as a bounded projection from $L^\infty(d\eta\,d\sigma)$ onto the subspace $\varphi h_\infty(\varphi)$.

Received 11.04.2013

REFERENCES

1. **Rubel L. A., Shields A. L.** The Second Duals of Certain Spaces of Analytic Functions. // J. Austral. Math. Soc., 1970, v. 11, p. 276–280.
2. **Shields A. L., Williams D. L.** Bounded Projections, Duality and Multipliers in Spaces of Harmonic Functions. // J. Reine Angew. Math., 1978, v. 299&300, p. 256–279.
3. **Petrosyan A.I.** Bounded Projectors in Spaces of Functions Holomorphic in the Unit Ball. // J. Cont. Math. Anal., 2011, v. 46, № 5, p. 264–272.
4. **Petrosyan A.I.** On Weighted Classes of Harmonic Functions in the Unit Ball of $\mathbf{R^n}$. // Complex Variables, 2005, v. 50, № 12, p. 953–966.
5. **Axler Sh., Bourdon P., Ramey W.** Harmonic Function Theory. New York: Springer-Verlag Inc., 2001.