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Mathematics

BOUNDED PROJECTORS ON L^p SPACES IN THE UNIT BALL

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The paper studies the linear operators depending on normal pair of weight functions $\{\varphi,\psi\}$ in the Banach spaces $L^p(B)$. Here B is the unit ball in the complex space $\mathbb{C}^n$. In particular, we study the question: for which values of p these operators are bounded projectors.

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Introduction. The paper is devoted to investigation of some linear operators depending on a normal pair of weight functions. The concept of a normal pair of weight functions was first introduced by Shields and Williams [1], and it appeared to be a convenient notion for statement of the estimates for integrals and for exposition of projectors in weight spaces.

Definition 1. A positive, continuous in $[0,1)$ function φ is called normal, if there are some constants $0<\varepsilon<k$ and $0\le r_0<1$ such that

$$\frac{\varphi(r)}{(1-r)^{\varepsilon}} \quad \text{decreases as} \quad r_0\le r<1 \quad \text{and} \quad \lim_{r\to 1^-}\frac{\varphi(r)}{(1-r)^{\varepsilon}}=0,$$
$$\frac{\varphi(r)}{(1-r)^{k}} \quad \text{increases as} \quad r_0\le r<1 \quad \text{and} \quad \lim_{r\to 1^-}\frac{\varphi(r)}{(1-r)^{k}}=\infty. \tag{1}$$

Note that k and ε are not uniquely determined by φ.

Definition 2. A pair of functions $\{\varphi,\psi\}$ is called a normal pair, if φ is normal and if, for some k satisfying (1), there exists $\alpha>k-1$ such that

$$\varphi(r)\psi(r)=\left(1-r^2\right)^{\alpha}, \qquad 0\le r<1. \tag{2}$$

Note that due to condition $\alpha>k-1$ the function ψ is integrable on the $[0,1)$. If $\alpha>k$, then also ψ with the suitable degrees $\alpha-k$ and $\alpha-\varepsilon$ is normal. The following notation is used throughout the paper: $\langle z,w\rangle=\sum_{k=1}^{n} z_k\overline{w}_k$ means the inner product of the points $z,w\in\mathbb{C}^n$ and $|z|=\sqrt{\langle z,z\rangle}$ the induced norm.

$B=\{z\in\mathbb{C}^n:\ |z|<1\}$ stands for the open unit ball in $\mathbb{C}^n$ and $S=\partial B=\{z\in \mathbb{C}^n:\ |z|=1\}$ for its boundary, which is the unit sphere in $\mathbb{C}^n$. $H(B)$ means the set of functions holomorphic in B. ν stands for the Lebesgue measure of the volume element of $\mathbb{C}^n$ normed by the condition $\nu(B)=1$. σ stands for the Lebesgue measure of the area element

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on S normed by the condition $\sigma(S) = 1$. $L^p(B)$ stands the set of all measurable functions f in B, for which

$$\|f\|_p = \left(\int_B |f(z)|^p \, d\nu(z)\right)^{\frac{1}{p}} < +\infty, \qquad 0 < p < \infty.$$

Spaces $A^p(\psi)$. Below was assumed that $\{\varphi, \psi\}$ is a normal pair. We continue φ and ψ to the whole B as $\varphi(z) = \varphi(|z|)$, $\psi(z) = \psi(|z|)$. For $0 < p < \infty$, $\alpha > -1$ consider the following spaces of functions holomorphic in B:

$$A^p(\psi) = \left\{ f \in H(B) : \ \|f\|_{p,\psi} = \left(\int_B |f(z)\psi(z)|^p \, d\nu(z)\right)^{\frac{1}{p}} < \infty \right\},$$

$$A^p_\alpha = \left\{ f \in H(B) : \ \|f\|_{p,\alpha} = \left(\int_B |f(z)|^p \, (1-|z|^2)^\alpha d\nu(z)\right)^{\frac{1}{p}} < \infty \right\}.$$

Thus, A^p_α is a special case of $A^p(\psi)$, if ψ is a power function: $\psi(z) = \left(1-|z|^2\right)^{\frac{\alpha}{p}}$.

Proposition 1. For $1 \leqslant p < \infty$, $A^p(\psi)$ are Banach spaces.

It is proved by the same method that in [2], where the case $p = 1$ is considered.

Now let the mappings

$$T_p : A^p(\psi) \mapsto L^p(B), \quad 1 \leqslant p < \infty,$$

be defined by the equalities $T_p(g) = \psi g$. Obviously these mappings are isometries. The ranges are closed subspaces of $L^p(B)$. We use the following notation for the ranges of these mappings

$$TA^p = T\left(A^p(\psi)\right).$$

Consider the following integral operator:

$$(Qf)(z) = \gamma_\alpha \int_B \frac{\psi(z)\varphi(w)}{\left(1-\langle z,w\rangle\right)^{n+1+\alpha}} f(w) \, d\nu(w), \tag{3}$$

where

$$\gamma_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

In [1] the boundedness of this operator in the space $L^1(B)$ is proved in the case $n = 1$. In [2] this result is generalised for arbitrary n, namely the following theorem has been proved:

Theorem A. Operator (3) is bounded in $L^1(B)$. Moreover, it is a bounded projector on the subspace TA^1.

The following natural problem arises: for which values of p operator Q is a bounded projector in $L^p(B)$. This paper is devoted to the solution of this problem. The main result is the Theorem 2. Note that the condition $p(k-\alpha) < 1$ for boundedness of operator Q is sufficient, but not necessary. This can be easily seen, as the number k in (1) can be choosen in many ways of normal function is ambiguous. In the case when weight functions are power functions, the specified condition is as well necessary. Let $\varphi(r) = (1-r)^b$, $\psi(r) = (1-r)^a$, $a + b = \alpha$. The following result has been proved in [3] (Theorem 2.10):

Theorem B. For two real numbers $a > -1$ and $b > 0$ we define two integral operators:

$$(Qf)(z) = \left(1-|z|^2\right)^a \int_B \frac{\left(1-|z|^2\right)^b}{\left(1-\langle z,w\rangle\right)^{n+1+a+b}} f(w) \, d\nu(w)$$

and

$$(\widetilde{Q}f)(z) = \left(1-|z|^2\right)^a \int_B \frac{\left(1-|z|^2\right)^b}{\left|1-\langle z,w\rangle\right|^{n+1+a+b}} f(w) \, d\nu(w).$$

For $1 \leqslant p < \infty$ the following conditions are equivalent:

a) Q is bounded in $L^p(B)$;

b) $\widetilde{Q}$ is bounded in $L^p(B)$;

c) $-pa < 1$.

Auxiliary Lemmas.

Lemma 1. For $\gamma > -1$ and $m > 1+\gamma$ we have

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\gamma}\,dr \leq C(1-\rho)^{1+\gamma-m}, \quad 0<\rho<1,$$

where the constant $C = C(m,\gamma)$ does not depend on ρ.

For the proof see [1].

Lemma 2. For a positive $c > 0$

$$\int_S \frac{d\sigma(\zeta)}{|1-\langle\zeta,z\rangle|^{n+c}} = O\left((1-|z|)^{-c}\right) \quad as \quad |z| \to 1^-.$$

For the proof see [3, Theorem 1.12].

Lemma 3. For

$$-(1+\varepsilon) < qs < \alpha - k \tag{4}$$

there exist a constant M_0 such that

$$\int_B \frac{\left(1-|w|^2\right)^{qs}\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(w) \leqslant M_0\,\frac{\left(1-|z|^2\right)^{qs}}{\psi(z)}.$$

Proof. According to the definition of a normal pair of functions, $\varphi(r)\psi(r) = (1-$ $-r^2)^{\alpha}$ for $0 \leqslant r < 1$, so, it will be sufficient to show that

$$\int_B \frac{\varphi(w)\,d\nu(w)}{(1-|w|^2)^{\beta/p}\,|1-\langle z,w\rangle|^{n+1+\alpha}} \leqslant M_0\,\frac{\varphi(z)}{(1-|z|^2)^{(\beta/p)+\alpha}}.$$

Let $w = r\zeta$, where $r = |w|$, $\zeta \in S$. As $\alpha+1>0$, then, due to Lemma 2, there exists a constant C such that

$$\int_S \frac{d\sigma(\zeta)}{|1-\langle z,r\zeta\rangle|^{n+1+\alpha}} \leqslant \frac{C}{(1-(|z|r)^2)^{\alpha+1}}.$$

Expressing the normed volume element $d\nu$ in polar coordinates, we get

$$d\nu(w) = 2n\,r^{2n-1}\,dr\,d\sigma(\zeta), \quad w = r\zeta, \text{ where } \zeta \in S$$

(see [3], Lemma 1.8), hence,

$$\begin{aligned}
&\int_B \frac{\left(1-|w|^2\right)^{qs}\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(w) = \\
&= 2n\int_0^1 \left(1-r^2\right)^{qs}\varphi(r)r^{2n-1}\left(\int_S \frac{d\sigma(\zeta)}{|1-\langle z,r\zeta\rangle|^{n+1+\alpha}}\right)dr \leqslant \qquad (5)\\
&\leqslant C\int_0^1 \frac{\left(1-r^2\right)^{qs}\varphi(r)}{(1-(|z|r)^2)^{\alpha+1}}\,dr \leqslant C_1\int_0^1 \frac{(1-r)^{qs}\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr
\end{aligned}$$

(in the last step we have replaced $(1-r^2)$ by $(1-r)$ and $\left(1-(|z|r)^2\right)$ by $(1-|z|r)$, so, we have changed C to some other constant C_1).

We divide the last integral into three parts:

$$\int_0^1 \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr = \int_0^{r_0} \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr +$$
$$+\int_{r_0}^{|z|} \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr + \int_{|z|}^1 \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr = I_1+I_2+I_3. \tag{6}$$

Obviously I_1 is bounded for all z. Therefore, there exists a constant C_2 such that

$$I_1 \leqslant C_2\varphi(z)\left(1-|z|^2\right)^{qs-\alpha} \leqslant M_0\,\frac{\left(1-|z|^2\right)^{qs}}{\psi(z)}, \tag{7}$$

here we have used the fact that $qs-\alpha+k<0$, so, the right-hand side of (7) is away zero, and also that $\varphi(z)\psi(z)=\left(1-|z|^2\right)^{\alpha}$. From the definition of normal function (1) we have

$$\frac{\varphi(r)}{(1-r)^k} \leqslant \frac{\varphi(z)}{(1-|z|)^k} \quad \text{for} \quad r_0<r\leqslant|z|. \tag{8}$$

Therefore,

$$I_2 = \int_{r_0}^{|z|} \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr = \int_{r_0}^{|z|} \frac{\varphi(r)\,(1-r)^{k+qs}}{(1-r)^k(1-|z|r)^{\alpha+1}}dr \leqslant \frac{\varphi(z)}{(1-|z|)^k}\int_{r_0}^{|z|} \frac{(1-r)^{k+qs}}{(1-|z|r)^{\alpha+1}}dr.$$

From (4) we get $k+qs>\varepsilon+qs>-1$, $\alpha+1>k+qs+1$, and, taking into account Lemma 1, we have

$$\int_{r_0}^{|z|} \frac{(1-r)^{k+qs}}{(1-|z|r)^{\alpha+1}}dr \leqslant \int_0^1 \frac{(1-r)^{k+qs}}{(1-|z|r)^{\alpha+1}}dr \leqslant C_3\,(1-|z|)^{k+qs-\alpha}$$

for some constant C_3. Therefore,

$$I_2 \leqslant \frac{\varphi(z)}{(1-|z|)^k}\int_{r_0}^{|z|} \frac{(1-r)^{k+qs}}{(1-|z|r)^{\alpha+1}}dr \leqslant C_3\frac{\varphi(z)}{(1-|z|)^k}\,(1-|z|)^{k+qs-\alpha} \leqslant$$
$$\leqslant C_4\varphi(z)\left(1-|z|^2\right)^{qs-\alpha} = C_4\,\frac{\left(1-|z|^2\right)^{qs}}{\psi(z)}.$$

To estimate I_3, we use the inequality.

$$\frac{\varphi(r)}{(1-r)^{\varepsilon}} \leqslant \frac{\varphi(z)}{(1-|z|)^{\varepsilon}} \quad \text{for} \quad |z|<r<1. \tag{9}$$

Then

$$I_3 = \int_{|z|}^1 \frac{(1-r)^{qs}\,\varphi(r)}{(1-|z|r)^{\alpha+1}}\,dr = \int_{|z|}^1 \frac{\varphi(r)\,(1-r)^{\varepsilon+qs}}{(1-r)^{\varepsilon}\,(1-|z|r)^{\alpha+1}}\,dr \leqslant$$
$$\leqslant \frac{\varphi(z)}{(1-|z|)^{\varepsilon}}\int_{|z|}^1 \frac{(1-r)^{\varepsilon+qs}}{(1-|z|r)^{\alpha+1}}\,dr \leqslant \frac{\varphi(z)}{(1-|z|)^{\varepsilon}}\int_0^1 \frac{(1-r)^{\varepsilon+qs}}{(1-|z|r)^{\alpha+1}}\,dr.$$

By (4), we have $\varepsilon+qs>-1$ and $\alpha+1>k+qs+1>\varepsilon+qs+1$, so, using Lemma1 we get

$$\int_0^1 \frac{(1-r)^{\varepsilon+qs}}{(1-|z|r)^{\alpha+1}}\,dr \leqslant C_5\,(1-|z|)^{\varepsilon+qs-\alpha},$$

hence,

$$\begin{aligned} I_3 &\leqslant \frac{\varphi(z)}{(1-|z|)^{\varepsilon}} \int_0^1 \frac{(1-r)^{\varepsilon+qs}}{(1-|z|r)^{\alpha+1}}\,dr \leqslant \\ &\leqslant C_5\,\varphi(z)\,(1-|z|)^{qs-\alpha} \leqslant C_6\,\varphi(z)\left(1-|z|^2\right)^{qs-\alpha}, \end{aligned} \tag{10}$$

with some constants C_5 and C_6. Combining the obtained results (5)–(10), we get that there exist a constant M_0 such that

$$\int_B \frac{\left(1-|w|^2\right)^{qs}\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(w) \leqslant M_0\,\varphi(z)\left(1-|z|^2\right)^{qs-\alpha} = M_0\,\frac{\left(1-|z|^2\right)^{qs}}{\psi(z)}$$

for $-(1+\varepsilon)<qs<\alpha-k$. The Lemma is proved. □

L e m m a 4. Let

$$k-\alpha-1<ps<\varepsilon. \tag{11}$$

Than there exists a constant M_1 such that

$$\int_B \frac{\left(1-|z|^2\right)^{ps}\psi(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(z) \leqslant M_1\,\frac{\left(1-|w|^2\right)^{ps}}{\varphi(w)}.$$

Proof. We have $\varphi(z)\psi(z)=\left(1-|z|^2\right)^{\alpha}$. Therefore, it is necessary to prove that

$$\int_B \frac{\left(1-|z|^2\right)^{ps+\alpha}}{\varphi(z)\,|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\nu(z) \leqslant M_1\,\frac{\left(1-|w|^2\right)^{ps}}{\varphi(w)}$$

for some constant M_1. As in the proof of the previous lemma, we pass to polar coordinates, apply the Lemma 2, split the obtained integral into three parts and estimate each part separately. Here we must take in account that instead of inequalities (8) and (9) it is necessary to use inequalities

$$\begin{aligned} \frac{(1-r)^{\varepsilon}}{\varphi(r)} &\leqslant \frac{(1-|w|)^{\varepsilon}}{\varphi(w)} \quad \text{for} \quad r_0<r\leqslant|w|, \\ \frac{(1-r)^{k}}{\varphi(r)} &\leqslant \frac{(1-|w|)^{k}}{\varphi(w)} \quad \text{for} \quad |w|<r<1. \end{aligned}$$

The remaining arguments are similar to the proof of the previous Lemma, with minor alterations, and we omit the details. □

The Main Result. The following Schur's test is a useful tool to prove the L^p- boundedness of operators.

T h e o r e m 1 . *(Schur's test)* Suppose (X,μ) is a measure space, $1<p<\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. For a nonnegative kernel $H(x,y)$ consider the integral operator

$$(Sf)(x)=\int_X H(x,y)f(y)\,d\mu(y).$$

If there exist a positive function h on X and a positive constant C such that

$$\int_X H(x,y)h(y)^q d\mu(y) \leqslant \mathrm{Ch}(x)^q$$

for almost all $x \in X$ and

$$\int_X H(x,y)h(x)^p d\mu(x) \leqslant \mathrm{Ch}(y)^p$$

for almost all $y \in X$, then the operator S is bounded in $L^p(X,\mu)$ with $\|S\| \leqslant C$.

For the proof see [4], (Theorem 1.8) or [3], (Theorem 2.9).

Theorem 2. For all p, $1 \leqslant p < \infty$, with

$$p(k-\alpha) < 1, \tag{12}$$

the integral operator

$$(Qf)(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_B \frac{\psi(z)\varphi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}} f(w)\, d\nu(w) \tag{13}$$

is a bounded projector in $L^p(B)$.

Proof. First of all we will prove boundedness Q. First consider the case $1 < p < \infty$. It is enough to prove boundedness of the following operator with positive kernel:

$$(Sf)(z) = \int_B \frac{\psi(z)\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} f(w)\, d\nu(w), \tag{14}$$

since from the boundedness (14) the boundedness of (13) follow immedietly. According to the Schur's test, it would be sufficient to find suitable function h, for which the inequalities

$$\int_B \frac{\psi(z)\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} h(w)^q d\nu(w) \leqslant Mh(z)^q,$$

$$\int_B \frac{\psi(z)\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} h(z)^p d\nu(z) \leqslant Mh(w)^p$$

are fulfilled. We look the function h in the following form: $h(z) = \left(1-|z|^2\right)^s$. That is, we need to prove that

$$\int_B \frac{\left(1-|w|^2\right)^{qs}\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}} d\nu(w) \leqslant M\frac{\left(1-|z|^2\right)^{qs}}{\psi(z)}, \tag{15}$$

and

$$\int_B \frac{\left(1-|z|^2\right)^{ps}\psi(z)}{|1-\langle z,w\rangle|^{n+1+\alpha}} d\nu(z) \leqslant M\frac{\left(1-|w|^2\right)^{ps}}{\varphi(w)} \tag{16}$$

for some s. According to Lemmas 3 and 4, the inequalities (15) and (16) are true under the conditions

$$-\frac{1+\varepsilon}{q} < s < \frac{\alpha-k}{q} \quad \text{and} \quad \frac{k-\alpha-1}{p} < s < \frac{\varepsilon}{p}$$

with $M = \max\{M_0, M_1\}$. It is only necessary to determine for which values of p the intersection of intervals

$$\left(-\frac{1+\varepsilon}{q},\ \frac{\alpha-k}{q}\right) \quad \text{and} \quad \left(\frac{k-\alpha-1}{p},\ \frac{\varepsilon}{p}\right)$$

is nonempty. Obviously $-\dfrac{1+\varepsilon}{q} < \dfrac{\varepsilon}{p}$. Therefore, indicated intersection is nonempty, if and only if $\dfrac{k-\alpha-1}{p} < \dfrac{\alpha-k}{q}$. As $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we receive $p(k-\alpha) < 1$. That is for $p(k-\alpha) < 1$ $(1 < p < \infty)$ the operator (14) is bounded in $L^p(B)$.

The statement of the Theorem for the case $p = 1$ follows from the Theorem A. Indeed, the condition (12) in this case is equivalent to condition $\alpha > k-1$ for normal pair $\{\varphi, \psi\}$ (see Definition 2).

Now we prove that P is projector. The range of Q,obviously belongs to TA^p. We show, that on TA^p the operator Q is identical. Let $f \in TA^p$. Then $f = \psi g$, where $g \in A^p(\psi)$. We have $Q(\psi g) = P_\alpha(g) = g$. Here P_α denote the integral operator (17), which is defined below. The equality $P_\alpha(g) = g$ follows from the fact that $A^p(\psi) \subset A^1(\psi) \subset A^1_\alpha$, and that in the space A^1_α the kernel $K_\alpha(z,w)$ is reproducing.

The Theorem is proved. □

Note that the case $n = 1$ of Theorem 2 is considered in [5].

Corollary 1. For all $1 \leqslant p < \infty$, $\alpha > 0$, the Bergman type integral operator

$$(P_\alpha f)(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_B \frac{\left(1-|w|^2\right)^\alpha}{\left(1-\langle z,w\rangle\right)^{n+1+\alpha}} f(w)\,d\nu(w) \tag{17}$$

is a bounded projektor in $L^p(B)$.

Proof. Consider the particular case when φ is power function: $\varphi(r) = (1-r)^\alpha$, where $\alpha > 0$ and $\psi(r) \equiv 1$. Then the corresponding operator Q has the form (17). For any given $1 \leqslant p < \infty$ is possible to find a number $k > \alpha$ satisfying to the condition (12) of Theorem 2.

Note that the statement of Corollary 1 follows also from Theorem B at $a = 0$, and also from Rudin–Forelli theorem [6].

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