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Mathematics

LOCAL EXISTENCE THEOREM FOR THE EQUATIONS OF MOTION OF VISCOUS LIQUID IN HÖLDER WEIGHT SPACES

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In this paper a proof of a local existence theorem for the equation of motion of viscous liquid in Hölder weight spaces is presented.

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Introduction. In this paper we present a proof of a local existence theorem for the equation of motion of viscous liquid in Hölder weight spaces

$$\begin{aligned} &\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\vec{v}=0;\quad \rho\frac{\partial\vec{v}}{\partial t}=\rho\cdot\vec{f}+\nabla(\lambda\cdot\nabla\cdot\vec{v})+2\nabla\cdot\mu D-\nabla p,\\ &\rho c_v\frac{\partial\theta}{\partial t}=\nabla\cdot\kappa\nabla\theta+(\nabla\cdot\vec{v})^2+2\mu D:D+(\rho^2E_\rho-E)\nabla\cdot\vec{v}. \end{aligned}\tag{1}$$

Here ρ is the density, $v=(v_1,v_2,v_3)$ is the velocity vector, $\vec{f}$ is a vector of external forces, $c_v=E_\theta=\dfrac{\partial E(\rho,\theta)}{\partial\theta}$ is the specific heat of the liquid which is a positive functions on ρ and θ, $E_\rho=\dfrac{\partial E(\rho,\theta)}{\partial\rho}$.

Pressure p, specific internal energy E, coefficients of viscosity λ, μ, and coefficient of heat conduction κ are given functions on the variables ρ, θ satisfying the conditions $\kappa,\mu>0$, $2\mu+\lambda>0$. Let ∇ be an operator of differentiation with respect to variables $x_i:\nabla=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$. For any function $a(x)$, vector $\vec{b}(x)$, and matrix $A(x)$ with elements $a_{ij}(x)$, $i,j=1,2,3$, we have:

$$\nabla a=\left(\frac{\partial a}{\partial x_1};\frac{\partial a}{\partial x_2},\frac{\partial a}{\partial x_3}\right);\ \nabla\cdot b=\frac{\partial b_1}{\partial x_1}+\frac{\partial b_2}{\partial x_2}+\frac{\partial b_3}{\partial x_3};\ (\vec{b}\cdot\nabla)a=b_1\frac{\partial a}{\partial x_1}+b_2\frac{\partial a}{\partial x_1}+$$

$$+b_3\frac{\partial a}{\partial x_1};\ \nabla\cdot A(x)=g,\quad g_i=\frac{\partial A_{1i}}{\partial x_1}+\frac{\partial A_{2i}}{\partial x_2}+\frac{\partial A_{3i}}{\partial x_3};\ \frac{df}{dt}=\frac{\partial f}{\partial t}+(\vec{v}\cdot\nabla f).$$

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Finally, $D(\vec{v})$ is the deformation tensor, i.e. a matrix with elements

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),\ i,j = 1,2,3,\ D:D = \sum_{i,j=1}^{3} D_{ij}D_{ij}.$$

All these functions depend on points (x_1, x_2, x_3) of the domain Ω filled by liquid.

The initial boundary value problem for the system (1) is considered. We suppose that the walls of the container of liquid are fixed and the usual adherence condition for velocity vector is satisfied

$$\vec{v}l_{\Gamma_T} = 0, \quad \theta|_{\Gamma_T} = \theta_1(x,t), \tag{2}$$

where $\Gamma_T = \partial\Omega \times [0,T]$. We suppose that the following initial conditions hold:

$$\rho(x,0) = \rho_0(x), \quad \vec{v}(x,0) = \vec{v}_0(x), \quad \theta(x,0) = \theta_0(x). \tag{3}$$

Now let recall the definition of Hölder weight space. By $C^s(\Omega)$ with any $s \geqslant 0$, we denote the space of functions, which are $[s]$ times continuously differentiable in the domain Ω and have the finite norm $|u|_\Omega^{(s)} = \sum_{|\alpha|<s} |D^\alpha u|_\Omega + [u]_\Omega^{(s)}$, where $|u|_\Omega = \sup_{x\in\Omega} |u(x)|$, $[u]_\Omega^{(s)} = \sum_{|\alpha|=s} |D^\alpha u|_\Omega$ and $[u]_\Omega^{(s)} = \sum_{|\alpha|=[s]} \sup_{x,y\in\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{s-[s]}}$ for integer and non-integer s respectively.

Let $C_{s,Q_T}^{l,l/2}$ with any non-integer $l > 0$ and $s \in [0,l)$ be a set of functions given in Q_T with the following finite norm

$$\begin{aligned} |u|_{s,Q_T}^{(l)} = \sup_{t<T} t^{\frac{l-s}{2}} [u]_{Q_t}^{(t)} + \sum_{s<2\alpha+|\gamma|<l} \sup_{t<T} t^{\frac{2\alpha+|\gamma|-s}{2}} |D_t^\alpha D_x^\gamma u|_\Omega + [u]_{Q_T}^{(s)} + \\ + \sum_{2\alpha+|\gamma|\leqslant s} \sup_{t<T} |D_t^\alpha D_x^\gamma u(x,t)|_\Omega, \end{aligned} \tag{4}$$

where $[u]_{Q_T}^{(s)} = \sup_{t<T} [u]_\Omega^{(s)} + \sum_{s-2<2a+\gamma\leqslant s} \sup_{x,t,\tau} \frac{|D_t^a D_x^\gamma u(x,t) - D_t^a D_x^\gamma u(y,t)|}{|t-\tau|^{\frac{s-2a-|\gamma|}{2}}}$ and $Q_t = \Omega \times (t/2, t)$. For $s = l$ this space matches with the space $C^{l,l/2}(Q_T)$. For $s < 0$ define the norm $C_s^{l,l/2}(Q_T)$ by the same formula (4) without the last two summands. In [1] the following result was obtained:

Theorem A [1].Let $\Omega \subset R^3$ be a bounded or unbounded domain. whose boundary is from the class $C^{2+\alpha}$, $\alpha \in (0,1)$. Let the function f and $\frac{\partial f}{\partial x_i}$ be from $C^{\alpha,\frac{\alpha}{2}}(Q_{t_0})$, $\rho \in C^{1+\alpha}(\Omega)$, $\vec{v}_0, \theta_0 \in C^{2+\alpha}(\Omega)$, $\theta_1 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_{t_0})$ and we have the relations:

$$0 < \theta' \leqslant \theta_0(x) \leqslant \theta'', \quad \theta' \leqslant \theta_1(x) \leqslant \theta'', \quad 0 < \rho' \leqslant \rho_0(x) \leqslant \rho'', \tag{5}$$

together with the following consistency conditions for any $x \in S$:

$$\vec{v}_0(x) = 0, \quad \theta_0(x) = \theta_1(x,0), \quad x \in S = \partial\Omega, \tag{6}$$

$$\begin{aligned} &\rho_0(x) f(x,0) + \nabla(\lambda(\rho_0,\theta_0)\nabla\cdot\vec{v}_0) + 2\nabla\mu(\rho_0,\theta_0)D(\vec{v}_0) - \nabla p(\rho_0,\theta_0) = 0, \\ &\nabla\kappa(\rho_0,\theta_0)\nabla\theta_0 + \lambda(\rho_0,\theta_0)(\nabla\cdot\vec{v}_0)^2 + 2\mu(\rho_0,\theta_0)D(\vec{v}_0) : D(\vec{v}_0) + \\ &+ \left(\rho_0^2 E_{p_0}(\rho_0,\theta_0) - P(\rho_0,\theta_0)\nabla\cdot\vec{v}_0\right) = \rho_0 C_{v_0}(\rho_0,\theta_0) \left.\frac{\partial\theta_1}{\partial t}\right|_{t=0}. \end{aligned} \tag{7}$$

We assume that $\lambda(\rho,\theta)$, $\mu(\rho,\theta)$, $P(\rho,\theta)$, $E(\rho,\theta)$ are defined for
$$\beta\rho' \leqslant \rho \leqslant \beta^{-1}\rho'', \quad \beta\theta' \leqslant \theta \leqslant \beta^{-1}\theta'', \quad \beta \in (0:1) \tag{8}$$
and belong to $C^{2+\alpha}$.

Then, the problem (1)–(3) have a unique solution $(\vec{v},\rho,\theta)$ determined on Q_{t_1} $(t_1 \leqslant t_0)$, $\vec{v} \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_1})$, $\theta \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{t_1})$, ρ, $\dfrac{\partial\rho}{\partial x_i}, \dfrac{\partial p}{\partial x_i} \in C^{\alpha,\frac{\alpha}{2}}(Q_{t_1})$.

In the present paper we extend the mentioned result for larger class of spaces $C_{1+\alpha}^{2+\alpha,1+\frac{\alpha}{2}}(Q_t)$, which elements may have derivatives and a singularity at $t=0$. This kind of generalization allows to prove solvability of the problem (1)–(3) with less smoothness requirements $\theta_0,\ v_0 \in C^{1+\alpha}(\Omega)$ and fewer conditions than in the works of Tani [1,2].

Some Preliminary Assumptions. Let a vector $\vec{u} \in C_{1+\alpha}^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$, where $Q_T = \Omega \times (0,T)$. We consider the one-parameter family of transformations
$$x = \xi + \int_0^t \vec{u}(\xi,\tau)\,d\tau = X(\xi,t) \tag{9}$$
of the domain Ω into the domain $\Omega_t \subset R^3$ with the boundary S_t.

By $U(t)$ we denote Jakob matrix of transformations (9) with entries
$$a^{kl} = \delta_{kl} + \int_0^t \frac{\partial u_k(\xi,\tau)}{\partial \xi_l}\,d\tau. \tag{10}$$
Assume $A = (U^*)^{-1}$ and denote its entries by a_{kl}. To guarantee the condition $\det U \neq 0$ we require the smallness conditions just as in [3,4] were done
$$|\vec{u}|_{1+\alpha,Q_{T_1}}^{(2+\alpha)}\left(T_1^{\frac{1+\alpha}{2}} + T_1^{\frac{1}{2}}\right) \leqslant \delta. \tag{11}$$
Then if $t \leqslant T_1$, $T_1 \leqslant T$, we have $1-3\delta-6\delta^2-6\delta^3 \leqslant \det U(\xi,t) \leqslant 1+3\delta+6\delta^2+6\delta^3$, and when $\delta \leqslant \dfrac{1}{8}$ we have $\dfrac{1}{2} \leqslant \det U(\xi,t) \leqslant \dfrac{3}{2}$.

We present the estimations of A, $A-I$.

Lemma 1. For any $a(\xi,t) \in C_{\alpha-1}^{\alpha,\frac{\alpha}{2}}(Q_{T_1})$, $b(\xi,t) \in C_{\alpha}^{1+\alpha,\frac{1+\alpha}{2}}(Q_{T_1})$ we have
$$\left|\int_0^t a(\xi,\tau)\,d\tau\right|_{1-\alpha,Q_{T_1}}^{\left(\alpha,\frac{\alpha}{2}\right)} \leqslant T_1^{\frac{1+\alpha}{2}}\,|a|_{1-\alpha,Q_{T_1}}^{(\alpha)} + T_1^{\frac{1}{2}}\,|a|_{1-\alpha,Q_{T_1}}^{(\alpha)}, \tag{12}$$
$$\left|\int_0^t b(\xi,\tau)\,d\tau\right|_{Q_{T_1}}^{\left(1+\alpha,\frac{\alpha}{2}\right)} \leqslant T_1\,|b|_{\alpha,Q_{T_1}}^{(1+\alpha)} + T_1^{\frac{1+\alpha}{2}}\,|b|_{\alpha,Q_{T_1}}^{(1+\alpha)} + T_1^{\frac{1}{2}}\,|b|_{\alpha,Q_{T_1}}^{\alpha+1}. \tag{13}$$

Lemma 2. For $\delta \leqslant 1/8$ the following inequalities hold:
$$|U-I|_{\alpha,Q_{T_1}}^{1+\alpha,\frac{1+\alpha}{2}} \leqslant 2\delta, \tag{14}$$
$$|A|_{\alpha,Q_{T_1}}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} \leqslant C_1, \tag{15}$$
$$|A-I|_{\alpha,Q_{T_1}}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} \leqslant C_2\,|u-I|_{\alpha,Q_{T_1}}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} \leqslant 2C_2\delta_1, \tag{16}$$
where C_1, C_2 are independent of δ.

L e m m a 3. Let the condition (11) be satisfied for vectors u and u'. Then for $t \in [0,T_1]$ the following inequalities hold:

$$\begin{aligned} |U-U'|_{\alpha,Q_t}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} &\leqslant C\left\{t^{\frac{1+\alpha}{2}}\,|\vec{u}-\vec{u}'|_{1+\alpha,Q_t}^{(2+\alpha)} + t^{\frac{1}{2}}\,|\vec{u}-\vec{u}'|_{1+\alpha,Q_t}^{(2+\alpha)}\right\} \\ &\leq C_3\cdot t^{\frac{1}{2}}\left|\vec{u}-\vec{u}'\right|_{1+\alpha,Q_t}^{(2+\alpha)}, \end{aligned} \tag{17}$$

$$\left|A-A'\right|_{\alpha,Q_t}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} \leqslant C_4 t^{\frac{1}{2}}\left|\vec{u}-\vec{u}'\right|_{1+\alpha,Q_t}^{2+\alpha}, \tag{18}$$

$$\left|\frac{\partial A}{\partial \xi_i}-\frac{\partial A'}{\partial \xi_i'}\right|_{\alpha,Q_t}^{\left(1+\alpha,\frac{1+\alpha}{2}\right)} \leqslant C_5 t^{\frac{1}{2}}\left|\vec{u}-\vec{u}'\right|_{1+\alpha,Q_t}^{2+\alpha}. \tag{19}$$

L e m m a 4. If $\vec{u},\vec{u}' \in C_{1+\alpha}^{2+\alpha}(Q_T)$, then the inequalities

$$\left|g_{ij}(\vec{u})\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_6 T\,|\vec{u}|_{1+\alpha,Q_T}^{(2+\alpha)}, \tag{20}$$

$$\left|g_{ij}(\vec{u})-g_{ij}(\vec{u}')\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_7\left|\vec{u}-\vec{u}'\right|_{1+\alpha,Q_T}^{(2+\alpha)} \tag{21}$$

hold for $g_{ij}=\int\limits_0^t a_{ij}(\vec{u})\dfrac{\partial u_i}{\partial \xi_j}\,d\tau$, where a_{ij} are the elements of the matrix A.

P r o o f. From (15) and (18), (20) and (21) are obtained respectively.

L e m m a 5. If $\vec{u},\vec{u}' \in C_{1+\alpha}^{2+\alpha}(Q_T)$, then the inequalities

$$\left|l_{ij}(\vec{u})\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_8 T\,|\vec{u}|_{1+\alpha,Q_T}^{(2+\alpha)}, \tag{22}$$

$$\left|l_{ij}(\vec{u})-l_{ij}(\vec{u}')\right|_{1-\alpha,Q_t}^{(\alpha)} \leqslant C_9 T\left|\vec{u}-\vec{u}'\right|_{1+\alpha,Q_T}^{(2+\alpha)} \tag{23}$$

hold for $l_{ij}(u)=\exp\Big(-\int\limits_0^t a_{ij}\dfrac{\partial u_i}{\partial \xi_j}\,d\tau\Big)$, where a_{ij} are the entries of the matrix A.

L e m m a 6. Let $\vec{u},\vec{u}' \in C_{1+\alpha}^{2+\alpha}(Q_T)$ and $r_{ij}=\rho l_{ij}(\vec{u})$, where $\rho_0(\xi)\in C^{1+\alpha}(\Omega)$, $0<\bar{\rho}_0 \leqslant \rho_0(\xi) \leqslant \bar{\bar{\rho}}_0$. Then the following inequalities hold:

$$\left|r_{ij}(\vec{u})\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_{10}T\,|\vec{u}|_{1+\alpha,Q_T}^{(2+\alpha)}, \qquad \left|r_{ij}(\vec{u})\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_{11}T\cdot|\vec{u}|_{1+\alpha,Q_T}^{(2+\alpha)}, \tag{24}$$

$$\left|r_{ij}(\vec{u})-r_{ij}(\vec{u}')\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_{12}T\left|\vec{u}-u'\right|_{1+\alpha,Q_T}^{(2+\alpha)}, \tag{25}$$

$$\left|\frac{1}{r_{ij}(\vec{u})}-\frac{1}{r_{ij}(\vec{u}')}\right|_{1-\alpha,Q_T}^{(\alpha)} \leqslant C_{13}T\left|\vec{u}-u'\right|_{1+\alpha,Q_T}^{(2+\alpha)}. \tag{26}$$

The proof of (24)–(26) is obtained from inequalities (21), (22).

Similar inequalities are true also for the function $\dfrac{\partial r_{ij}}{\partial \xi_i}$.

Existence of Local Solutions for Initial Boundary Value. Let us switch from Euler coordinates to Lagrange coordinates in the system (1)

$$x=\xi+\int\limits_0^t \vec{u}(\xi,\tau)\,d\tau \equiv X(\xi,t). \tag{27}$$

In these coordinates the system (1) can be rewritten as

$$\frac{\partial r}{\partial t}+r\nabla_{\vec{u}}\cdot u=0, \tag{28}$$

$$r\frac{\partial \vec{u}}{\partial t}-\nabla_{\vec{u}}(\lambda\nabla_{\vec{u}}\cdot\vec{u})-2\nabla_{\vec{u}}\cdot\mu D_{\vec{u}}(\vec{u})-\nabla_{\vec{u}}P=0, \tag{29}$$

$$rc_v\cdot\frac{\partial T}{\partial t}-\nabla_{\vec{u}}\cdot\kappa\nabla_{\vec{u}}T-(\lambda\nabla_{\vec{u}}\cdot\vec{u})^2-2\mu D_{\vec{u}}(\vec{u}):D_{\vec{u}}(\vec{u})-F(r,T)\nabla_{\vec{u}}\cdot\vec{u}=0. \tag{30}$$

Here $r(\xi,t)$, $T(\xi,t)$ are the density and the temperature in Lagrange coordinates, $F(r,T)=r^2\dfrac{\partial E}{\partial r}-p(r,T)$, $D_{\vec{u}}(\vec{u})$ are matrices with the entries

$$\sum_{m=1}^{3}\left(a_{im}\frac{\partial u_j}{\partial\xi_m}+a_{jm}\frac{\partial u_i}{\partial\xi_m}\right),\quad i,j=1,2,3,$$

$$\nabla_u=A\nabla=\left\{\sum_{m=1}^{3}a_{1m}\frac{\partial}{\partial\xi_m},\sum_{m=1}^{3}a_{2m}\frac{\partial}{\partial\xi_m},\sum_{m=1}^{3}a_{3m}\frac{\partial}{\partial\xi_m},\right\},$$

a_{ij} are the entries of the matrix $A=\left(X'^{-1}\right)^T$, where X' is the Jacob transformation of the matrix X, i.e. $X'_{ik}=\dfrac{\partial x_i}{\partial\xi_k}=\delta_{ik}+\displaystyle\int_0^t\frac{\partial u_i}{\partial\xi_k}d\tau$, T is the sign of transposition.

So, a_{ij} are rational functions on X'_{ik}. Let us integrate Eq. (28) and consider the whole problem as a problem of determining $\vec{u}$, T, which satisfy the initial boundary conditions

$$\vec{u}|_{D_T}=0,\quad T|_{D_T}=0;\ \vec{u}(\xi,0)=\vec{v}_0(\xi),\quad T(\xi,0)=\theta_0(\xi), \tag{31}$$

and the systems (29) and (30) with

$$r(\xi,t)=\rho_0(\xi)\exp\left(-\int_0^t\nabla_{\vec{u}}\cdot\vec{u}(\xi,\tau)c(\tau)\right). \tag{32}$$

Let $\vec{u}'$, T' be functions given in $D_T=\Omega\times(0,T)$, which satisfy (31) (it is easy to see that such functions exist). Let $\hat{X}=\xi+\displaystyle\int_0^t\vec{u}'(\xi,\tau)d\tau$, $A'=\left(\hat{X}'^{-1}\right)^T$, $r=\rho_0\exp\left(-\displaystyle\int_0^t\nabla_{\vec{u}}\cdot\vec{u}\,d\tau\right)$. Consider an auxiliary initial boundary value problem

$$\begin{aligned}&\frac{\partial\vec{u}}{\partial t}-\frac{1}{r'}\nabla_{\vec{u}'}(\nabla_{\vec{u}'}\cdot\vec{u})-\frac{2}{r'}\nabla_{\vec{u}'}\cdot\mu' D_{\vec{u}'}(\vec{u})+\frac{p'_{T'}}{r'}\nabla_{\vec{u}'}\cdot T=\frac{-p'_{T'}}{r'}\cdot\nabla_{\vec{u}'}r'\\&\frac{\partial T}{\partial t}-\frac{1}{r'c'_v}\nabla_{\vec{u}}\cdot k\nabla_{\vec{u}'}T-\frac{\lambda}{r'c'_v}(\nabla_{u'}\cdot\vec{u})(\nabla_{\vec{u}'}\cdot\vec{u})-\frac{2\mu'}{r'c'_v}D(\vec{u}'):D_{\vec{u}'}(\vec{u})\\&-\frac{1}{r'c'_v}F'\nabla_{\vec{u}'}\cdot\vec{u}=0.\end{aligned} \tag{33}$$

$$\vec{u}\Big|_{\partial D_T}=0,\quad T\Big|_{\partial D_T}=0\ \text{ and }\ \vec{u}(\xi,0)=\vec{v}_0(x),\quad T(\xi,0)=\theta_0(\xi), \tag{34}$$

where $\lambda'=\lambda(r',T')$, $p'=p(r',T')$ and so on, and $D_{\vec{u}'}(\vec{u})$ is a matrix with entries

$$\sum_{m=1}^{n}\left(a'_{im}\frac{\partial u_j}{\partial \xi_m}+a'_{jm}\frac{\partial u_i}{\partial \xi_m}\right), \quad \nabla_{\vec{u}'}=A'\nabla.$$

As it was shown in the work [1], the system (33) is of parabolic type. If we introduce vectors $U=\{\vec{u},T\}$, $U'=\{\vec{u}',T'\}$, $U_1=\{0,\theta_1\}$, $U_0=\{\vec{v}_0,\theta_0\}$, $H(U')=\left\{\frac{-p_{r'}}{r'}\nabla_{\vec{u}}r',0\right\}$, then we can rewrite (33), (34) in the form

$$\mathcal{L}(U')U=H(U'), \quad U|_{\Gamma_t}=U_1^0, \quad U|_{t=0}=U_0, \tag{35}$$

where $\mathcal{L}(U')$ is the matrix of differential operators of the left part of (33). Assuming $U'=U$ in the initial non-linear problem (35), we obtain

$$\mathcal{L}(U)U=H(U), \quad U|_{\Gamma_t}=U_1^0, \quad U|_{t=0}=U_0. \tag{36}$$

Theorem. Let $\Omega\subset R^3$ be a bounded or unbounded domain with boundary from the class $C^{2+\alpha}$, $\alpha\in(0,1)$. Let the functions $\rho_0,\theta_0,\vec{v}_0\in C^{1+\alpha}(\Omega)$, $\theta_1\in C^{2+\alpha}_{1+\alpha}(\Gamma_t)$ satisfy the relations (5), (6). Assume the function $\lambda(\rho,\theta)$, $\mu(\rho,\theta)$, $p(\rho,\theta)$, $E(\rho,\theta)$ are given in the domain determined by the conditions (5) and have bounded partial derivatives of the first and the second orders. Then the problem (1)–(3) has the unique solution (v,ρ,θ) determined on Q_t, $t\leqslant t_0$, $\vec{v},\theta\in C^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha}(Q_{t'})$ and $\rho,\dfrac{\partial\rho}{\partial x_i},\dfrac{\partial\rho}{\partial t}\in C^{\alpha,\frac{\alpha}{2}}_{\alpha-1}(Q_{t'})$, and consistency conditions (6) are satisfied.

Proof. As we have already seen, the problem (1)–(3) can be written in the form of (36). Now we will reduce the problem (36) to an equation of the form $V=BV$ in the Banach space $V\subset C^{2+\alpha}_{1+\alpha}(Q_t)$ of the vectors vanishing on the boundary Γ_t, when $t=0$.

Consider the problem

$$\mathcal{L}(U_0)U_1=H(U_0), \quad U_1|_{\Gamma_t}=U_1^0, \quad U_1|_{t=0}=U_0, \tag{37}$$

where U_1 is 4-component vector, $U_0\in C^{1+\alpha}(\Omega)$, $U_1^0\in C^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha}(\Gamma_t)$, $U_1(\xi,0)=U_0(\xi)$. From the results of the work [5] we know that the problem (37) has a unique solution in the class of functions $C^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha}(Q_t)$, for which the following evaluation is true

$$|U_1|^{2+\alpha}_{1+\alpha,Q_t}\leqslant C\left\{\left|U_1^0\right|^{2+\alpha}_{1+\alpha,\Gamma_t}+|U_0|^{(1+\alpha)}_{\Omega}\right\}. \tag{38}$$

After subtracting (37) from (36) and putting $V=U-U_1$, we will obtain

$$\begin{aligned}&\mathcal{L}(U_1)V=[\mathcal{L}(U_1)-\mathcal{L}(U_1+V)](U_1+V)+[H(U_1+V)-H(U_1)]+\\&+[\mathcal{L}(U_0)-\mathcal{L}(U_1)]U_1+[H(U_1)-H(U_0)], \qquad V\big|_{\Gamma_t}=0, \quad V\big|_{t=0}=0.\end{aligned}$$

This problem is equivalent to the equation

$$\begin{aligned}V=R(\tau)\{\mathcal{L}(U_1)-\mathcal{L}(U_1+V)\}(U_1+V)+[H(U_1+V)-H(U_1)]+\\+[\mathcal{L}(U_0)-\mathcal{L}(U_1)]U_1+[H(U_1)-H(U_0)]=B_\tau V=B_1 v,\end{aligned} \tag{39}$$

where $R(\tau)$ is an operator, which juxtaposes the vector $F\in C^{\alpha}_{\alpha-1}(Q_T)$ with the solution $v\in \overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{1+t,Q_\Gamma}$ of the mixed problem of parabolic system differential equations:

$$\mathcal{L}(U_1)V=F,\ V\big|_{\Gamma_t}=0, V\big|_{t=0}=0, \ \text{where}\ C^{2+\alpha}_{1+\alpha,Q_t}=\{v: v\in C^{2+\alpha}_{1+\alpha,Q_t}, v|_{\Gamma_t}=0\}. \tag{40}$$

The existence of bounded operator $R(\tau)$ in Hölder weight spaces is shown in the work [5]. Since $R(\tau)$ is bounded, if τ satisfies the condition

$$|U|^{2+\alpha}_{1+\alpha,Q_t}(\tau+\tau^{\frac{1-\alpha}{2}})\leqslant\delta \tag{41}$$

with a small $\delta>0$, we obtain

$$|V|^{2+\alpha}_{1+\alpha,Q_\tau}\leqslant C\|R(\tau)\|\left\{|[\mathcal{L}(U_1)-L(U_1+V)](U_1+V)|^{\alpha}_{1-\alpha,Q_\tau}+\right.$$
$$\left.|H(U_1+V)-H(U_1)|^{\alpha}_{\alpha-1,Q_\tau}+|[\mathcal{L}(U_0)-\mathcal{L}(U_1)]|^{\alpha}_{\alpha-1,Q_\tau}|H(U_1)-H(U_0)|^{\alpha}_{\alpha-1,Q_\tau}\right\}.$$

Thus, if U_1, U_1+V, U_1+W satisfy the condition (41),then $V(\xi,0)=W(\xi,0)=0$ and by Lemmas 1–5, we obtain the inequalities

$$|B_\tau V|^{2+\alpha}_{1+\alpha,Q_\tau}\leqslant\|R(\tau)\|\tau^{\gamma}\left\{c_1|V|^{2+\alpha}_{1-\alpha,Q_\tau}+c_2|U_0-U_1|^{2+\alpha}_{1+\alpha,Q_\tau}\right\}, \tag{42}$$

$$|B_\tau V-B_\tau W|^{2+\alpha}_{1+\alpha,Q_\tau}\leqslant\|R(\tau)\|c_3\tau^{\gamma}|V-W|^{2+\alpha}_{1+\alpha,Q_\tau}, \tag{43}$$

where $\gamma>0$ and the constants c_1, c_2, c_3 do not depend on τ, V, W. Chose the numbers $t_1>0$ and $\eta>0$ such that

$$|U_0|^{2+\alpha}_{1+\alpha,Q_t}\left(t_1+t_2^{\frac{1-\alpha}{2}}\right)\leqslant\delta,\ |U_1|^{2+\alpha}_{1+\alpha,Q_t}\left(t_1+t_2^{\frac{1-\alpha}{2}}\right)\leqslant\frac{\delta}{2},\ \|R(t_1)\|c_3t_1^{\gamma},$$
$$\|R(t_1)\|c_1t_1^{\gamma}\leqslant\frac{1}{2},\ 2c_2\|R(t_1)\|\ |U_1-U_0|^{2+\alpha}_{1+\alpha,Q_t}\cdot t_1^{\gamma}\leqslant\eta\leqslant\delta/2\left(t_1+t_2^{\frac{1-\alpha}{2}}\right)$$

denote by K_η the set of vectors satisfying conditions $|V|^{2+\alpha}_{1+\alpha,Q_{t_1}}\leq\eta$, from $\overset{\circ}{C}{}^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha,Q_{t'}}$. Since the operator B_{t_1} maps set K_η into itself, the Eq. (40) is solvable in K_η, and for any $V,W\in K_\eta$ we have $|B_{t_1}V-B_{t_1}W|^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha,Q_{t_1}}\leqslant c_4|V-W|^{2+\alpha,1+\frac{\alpha}{2}}_{1+\alpha,Q_{t-1}}$, $c_4<1$.

So, the solution of the problem (33)–(34) is obtained in the class of vectors from

$$C^{2+\alpha}_{1+\alpha}(Q_{t_1}),\quad T\in\left[\beta\bar{\theta}_0,\beta^{-1}\bar{\bar{\theta}}_0\right],\quad r\in\left[\beta\bar{\rho}_0,\beta^{-1}\bar{\bar{\rho}}_0\right].$$

The question on solvability of above mentioned problem on an arbitrary interval $(0,t_1)$ is still open.

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REFERENCES

1. **Tani A.** The Initial Value Problem for the Equation of Motion of General Fluid with Som Slip Boundary Condition. Keio Math. Sem. Rep., 1983, № 8, p. 77–83.
2. **Tani A.** Global Solution to the Conserved Phase Field Equations of Penrose–Fife Type. // Commun. Appl. Anal., 2012, v. 16, № 4, p. 703–706.
3. **Solonnikov V.A., Kazhikhov A.V.** Existence Theorem for the Equations of Motion of a Compressible Viscous Fluid. // Ann. Rev. Fluid. Mech., 1981, v. 13, p. 79–95.
4. **Solonnikov V.A.** Solvability of the Problem of the Motion of a Viscous Incompressible Fluid that Bounded by Free Surface. // Izv. Akad. Nauk SSSR. Mat., 1977, v. 41, № 6, p. 1388–1424 (in Russian).
5. **Solonnikov V.A., Khachatryan A.G.** Estimates of Solutions of Parabolic Initial Boundary Value Problems in Weighted Hölder Norms. // Trudi MNAN, 1980, v. 147, p. 147–155 (in Russian).