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Mathematics

ON MINIMAL COSET COVERING OF SOLUTIONS OF A BOOLEAN EQUATION

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For the equation $x_1x_2\dots x_n + x_{n+1}x_{n+2}\dots x_{2n} + x_{2n+1}x_{2n+2}\dots x_{3n} = 1$ over the finite field F_2 we estimate the minimal number of systems of linear equations over the same field such that the union of their solutions exactly coincides with the set of solutions of the equation. We prove in this article that the number in the question is not greater than $9n^{\log_2 3} + 4$.

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Introduction. Let F_2 be the finite field of 2 elements and F_2^n be an n-dimensional linear space over F_2. A **coset** of a linear subspace L in F_2^n is a translation of L, i.e. a set $\alpha + L \equiv \{\alpha + x \mid x \in L\}$ for some $\alpha \in F_2^n$. It is known that any k-dimensional coset in F_2^n can be represented as a set of solutions of a certain system of linear equations over F_2 of rank $n-k$ and vice versa.

Let A be a set of vectors in F_2^n. We say that a set of cosets $\{L_1, L_2, \dots, L_k\}$ is a covering for the set A, if and only if $L_i \subseteq A$ for $1 \le i \le k$ and $A = \bigcup_{i=1}^k L_i$. The length of the covering is the number of its cosets.

The purpose of this article is to estimate the minimal number of cosets in F_2^{3n} covering the set of all solutions of the equation

$$x_1x_2\dots x_n + x_{n+1}x_{n+2}\dots x_{2n} + x_{2n+1}x_{2n+2}\dots x_{3n} = 1. \tag{1}$$

The problem of the shortest or minimal coset covering of subsets in finite fields was introduced and investigated in [1–3]. It is clear that the coset in F_2^{3n} that corresponds to the system below is a subset of the set of solutions of the Eq. (1)

$$\begin{cases} x_i = 1, \quad i = 1, 2, \dots, n, \\ x_{n+1} = 0, \\ x_{2n+1} = 0. \end{cases}$$

One can construct a covering with $3n^2$ such cosets and one coset containing $(1, 1, \dots, 1)$ vector. The following theorem shows that a covering with an essentially smaller length exists.

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Theorem 1. Eq. (1) has a covering with a length $\leq 9n^{\log_2 3}+4$.

Blocking Sets. We employ the notion of a blocking set (see [4,5]). k**-blocking set** in F_2^n is a subset in F_2^n that has at least one common vector with each k-dimensional coset. Any 1-blocking set must contain at least 2^n-1 vectors, since any pair of vectors forms a coset of dimension 1.

Let $e_1, e_2, \dots, e_n$ be a standard basis in F_2^n. Define B by

$$\{0, e_1, e_2, \dots, e_n\} \cup \{e_i+e_j \mid 1 \leq i < j \leq n\}.$$

We are going to show that B is a $(n-2)$-blocking set. Let L be a coset of dimension $n-2$. If it contains the 0 vector then, B intersects with L, otherwise L coincides with the set of solutions of a linear system over F_2 of the form

$$\begin{cases} \alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n=1, \\ \beta_1x_1+\beta_2x_2+\ldots+\beta_nx_n=1, \end{cases}$$

where the equations are linearly independent (if the right part of one of the equations is 0 we replace that equation with a sum of the initial two equations).

If $\alpha_i=\beta_i=1$ for some i, then $e_i \in L$. Otherwise $i \neq j$ exist such that $\alpha_i=1, \beta_i=0, \alpha_j=0, \beta_j=1$. In this case $e_i+e_j \in L$. Hence, B is a $(n-2)$-blocking set containing $1+n+\dfrac{n(n-1)}{2}=\dfrac{n^2+n}{2}+1=O(n^2)$ vectors.

Theorem 2. The number of vectors in a minimal $(n-2)$-blocking set in $F_2^n \backslash \{0\}$ is not greater than $3n^{\log_2 3}$.

Proof. We construct a $(n-2)$-blocking set for $F_2^n \backslash \{0\}$ containing not more than $3n^{\log_2 3}$ vectors.

Let n be a power of 2. We construct the sets $F(n)$ recursively, so that $F(n)$ is a $(n-2)$-blocking set for $F_2^n \backslash \{0\}$, i.e. $F(n)$ has a common vector with each coset of dimension $n-2$ that does not contain the 0 vector.

Define $F(2)=\{(0,1),(1,0),(1,1)\}$. Let $F(n)$ be a $(n-2)$-blocking set for $F_2^n \backslash \{0\}$. We construct $F(2n)$ as follows.

Each vector $(\alpha_1, \alpha_2, \dots, \alpha_n) \in F(n)$ generates exactly 3 different vectors in $F(2n)$: $(\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_1, \alpha_2, \dots, \alpha_n)$, $(\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots, 0)$ and $(0, 0, \dots, 0, \alpha_1, \alpha_2, \dots, \alpha_n)$.

Hence, $|F(2n)|=3|F(n)|$, where $|A|$ stands for the number of elements in A. Now we verify that $F(2n)$ is a $(2n-2)$-blocking set for $F_2^{2n} \backslash \{0\}$. Let L be a nonzero coset of dimension $2n-2$ in $F_2^{2n} \backslash \{0\}$. As shown above L coincides with the set of solutions of the following linear system over F_2 consisting of the following equations:

$$\begin{cases} \alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n+\alpha_{n+1}x_{n+1}+\ldots+\alpha_{2n}x_{2n}=1, \\ \beta_1x_1+\beta_2x_2+\ldots+\beta_nx_n+\beta_{n+1}x_{n+1}+\ldots+\beta_{2n}x_{2n}=1. \end{cases} \tag{2}$$

Define $A_1=(\alpha_1, \dots, \alpha_n)$, $A_2=(\alpha_{n+1}, \dots, \alpha_{2n})$, $B_1=(\beta_1, \dots, \beta_n)$ and $B_2=(\beta_{n+1}, \dots, \beta_{2n})$. Obviously, both A_1 and A_2 or both B_1 and B_2 cannot be 0 vectors. We consider the following 3 cases:

1) A_1 and B_1 are both nonzero vectors; 2) A_2 and B_2 are both nonzero vectors;

3a) A_1, B_2 are nonzero and A_2, B_1 are zero vectors;

3b) A_2, B_1 are nonzero and A_1, B_2 are zero vectors.

In the case 1) consider the system

$$\begin{cases} \alpha_1 x_1 + \ldots + \alpha_n x_n = 1, \\ \beta_1 x_1 + \ldots + \beta_n x_n = 1. \end{cases}$$

Solutions of this system form a $(n-2)$-dimensional coset in F_2^n. Therefore, one of the solutions, say $(x_1^0, \ldots, x_n^0)$, belongs to $F(n)$. Then $(x_1^0, \ldots, x_n^0)$ appended with n zeros gives a solution for the System (2), which belongs to $F(2n)$. We deal with the case 2) in a similar way.

In the case 3a) consider the system

$$\begin{cases} \alpha_1 x_1 + \ldots + \alpha_n x_n = 1, \\ \beta_{n+1} x_1 + \ldots + \beta_{2n} x_n = 1. \end{cases} \tag{3}$$

Solutions of this system form a $(n-2)$-dimensional coset in F_2^n. Therefore, one of the solutions, say $(x_1^0, \ldots, x_n^0)$, belongs to $F(n)$. Then $(x_1^0, \ldots, x_n^0, x_1^0, \ldots, x_n^0)$ gives a solution for the System (3), which belongs to F_2^n. We deal with the case 3b) in a similar way.

Now we claim that $F(2n)$ is a $(2n-2)$-blocking set for $F_2^{2n} \backslash \{0\}$.

We have that $|F(2)| = 3$, $|F(2n)| = 3|F(n)|$; therefore for $n = 2^k$ it follows that $|F(n)| = 3^k = 3^{\log_2 n} = n^{\log_2 3}$.

Now let us consider the general case when n is arbitrary.

It is easy to verify that the number of elements in a minimal $(n-2)$-blocking set for $F_2^n \backslash \{0\}$ is a nondecreasing function depending on n (dropping the last coordinate in all vectors and forming a $(n-2)$-blocking set for $F_2^n \backslash \{0\}$ gives a $(n-3)$-blocking set for $F_2^{n-1} \backslash \{0\}$). If $2^k < n \le 2^{k+1}$, then the number of elements in a minimal $(n-2)$-blocking set for $F_2^n \backslash \{0\}$ is not greater than the number of elements in a minimal $(2^{k+1}-2)$-blocking set for $F_2^{2^{k+1}} \backslash \{0\}$, which is not greater than 3^{k+1}. Since $k < \log_2 n$, we have $3^{k+1} < 3 \cdot 3^{\log_2 n} = 3n^{\log_2 3}$. □

Corollary. The number of vectors in a minimal $(n-2)$-blocking set in F_2^n is not greater than $3n^{\log_2 3} + 1$.

We can get $(n-2)$-blocking set in F_2^n by adding $(0, 0, \ldots, 0)$ vector to $(n-2)$-blocking for $F_2^n \backslash \{0\}$.

Proof of Theorem 1. We start with some technical propositions that can be easily verified.

For $A \subseteq F_2^n$ and $B \subseteq F_2^m$ we define

$A \times B = \{(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) | (\alpha_1, \ldots, \alpha_n) \in A, (\beta_1, \ldots, \beta_m) \in B\}$.

Proposition 1. Let A and B be cosets in F_2^n and F_2^m respectively. Then $A \times B$ is a coset in F_2^{n+m} and $\dim(A \times B) = \dim A + \dim B$.

Proposition 2. Let cosets $L_1, \ldots, L_k$ form a covering for the set of all solutions of the equation $x_1 x_2 \ldots x_n = 0$ in F_2^n, such that any pair of different solutions belongs to at least one of cosets $L_i, i = 1, \ldots, n$. Then cosets $L_1 \times L_1, L_2 \times L_2, \ldots, L_k \times L_k$ form a covering for the set of all solutions except $x_i = 1, i = 1, \ldots 2n$, in F_2^{2n} for the equation $x_1 x_2 \ldots x_n + x_{n+1} x_{n+2} \ldots x_{2n} = 0$.

Proposition 3. Let cosets $L_1, \ldots, L_k$ form a covering for the set of all solutions of the equation $x_1 x_2 \ldots x_n = 0$ in F_2^n. Then there exists a coset covering for the same set consisting of $n-1$-dimensional cosets $H_1,\dots,H_k$, such that $L_i \subseteq H_i$ for $i=1,2,\dots,k$.

Proof. This immediately follows from the fact that a system of linear equations defining the coset L_i must contain at least one equation, which cannot be satisfied with the vector $(1,1,\dots,1)$.

This single equation defines the $(n-1)$-dimensional coset H_i.

Proposition 4. For any pair of vectors from $F_2^n \setminus \{(1,1,\dots,1)\}$ there exists a $(n-1)$-dimensional coset that contains both vectors and does not contain the vector $(1,1,\dots,1)$.

Starting from this point we will refer to $(n-1)$-dimensional cosets as hyperplanes.

Proposition 5. A hyperplane covering of length k for the set $F_2^n \setminus \{(1,1,\dots,1)\}$ such that any pair of vectors in $F_2^n \setminus \{(1,1,\dots,1)\}$ belongs to at least one of the covering hyperplanes exists, if and only if there exists an analogous covering for $F_2^n \setminus \{(0,0,\dots,0)\}$.

Proof. This becomes obvious, if one notes that $F_2^n \setminus \{(1,1,\dots,1)\}$ transforms into $F_2^n \setminus \{(0,0,\dots,0)\}$, the vector $(1,1,\dots,1)$ into $(0,0,\dots,0)$ and a hyperplane not containing $(1,1,\dots,1)$ into a hyperplane not containing $(0,0,\dots,0)$ when the vector $(1,1,\dots,1)$ is added to all vectors in F_2^n.

Lemma. The existence of a minimal hyperplane covering for $F_2^n \setminus \{(0,0,\dots,0)\}$ such that any pair of vectors in $F_2^n \setminus \{(0,0,\dots,0)\}$ belongs to at least one of the covering hyperplanes is equivalent to the existence of a minimal $(n-2)$-blocking set for $F_2^n \setminus \{(0,0,\dots,0)\}$.

Proof. Let $H_1,\dots,H_k$ be a covering by hyperplanes for $F_2^n \setminus \{(0,0,\dots,0)\}$ such that any pair of vectors in $F_2^n \setminus \{(0,0,\dots,0)\}$ belongs to at least one of the hyperplanes. Let H_i be defined by an equation $\alpha_1^i x_1 + \ldots + \alpha_n^i x_n = 1$, $i=1,2,\dots,k$. Define $H = \{(\alpha_1^i,\dots,\alpha_n^i) | i=1,2,\dots,k\}$. We prove that H is a $(n-2)$-blocking set for $F_2^n \setminus \{(0,0,\dots,0)\}$. Indeed, let a $(n-2)$-dimensional coset in $F_2^n \setminus \{(0,0,\dots,0)\}$ is defined by a system

$$\begin{cases} \beta_1 x_1 + \ldots + \beta_n x_n = 1, \\ \gamma_1 x_1 + \ldots + \gamma_n x_n = 1. \end{cases} \tag{4}$$

For a pair of vectors $(\beta_1,\dots,\beta_n)$ and $(\gamma_1,\dots,\gamma_n)$ there exists a hyperplane that covers both vectors, i.e. for some i we have $\alpha_1^i \beta_1 + \ldots + \alpha_n^i \beta_n = 1$ and $\alpha_1^i \gamma_1 + \ldots + \alpha_n^i \gamma_n = 1$, i.e. the vector $(\alpha_1^i,\dots,\alpha_n^i)$ is the requested common vector for the coset and H.

Let us now $H = \{(\alpha_1^i,\dots,\alpha_n^i) | i=1,2,\dots,k\}$ be a $(n-2)$-blocking set for $F_2^n \setminus \{(0,0,\dots,0)\}$. Denote by H_i the hyperplane in $F_2^n \setminus \{(0,0,\dots,0)\}$ defined by an equation $\alpha_1^i x_1 + \ldots + \alpha_n^i x_n = 1, i=1,\dots,k$. We prove that $H_1,\dots,H_k$ form a covering for $F_2^n \setminus \{(0,0,\dots,0)\}$ such that any pair of vectors in $F_2^n \setminus \{(0,0,\dots,0)\}$ belongs to at least one of the hyperplanes. Indeed, let $(\beta_1,\dots,\beta_n)$ and $(\gamma_1,\dots,\gamma_n)$ be different vectors from $F_2^n \setminus \{(0,0,\dots,0)\}$. Consider a $(n-2)$-dimensional coset in $F_2^n \setminus \{(0,0,\dots,0)\}$ defined by the System (4). There exists a common vector for H and this coset, that means for some i we have $\alpha_1^i \beta_1 + \ldots + \alpha_n^i \beta_n = 1$ and

$\alpha_1^i\gamma_1+\ldots+\alpha_n^i\gamma_n=1$. This implies that both vectors $(\beta_1,\ldots,\beta_n)$ and $(\gamma_1,\ldots,\gamma_n)$ are covered by H_i. □

Finally, we complete the proof of Theorem 1. We split the set of all solutions of the Eq. (1) into four disjoint subsets:

$A=\{(\alpha_1,\ldots,\alpha_{3n})|\alpha_1\ldots\alpha_n=1,\alpha_{n+1}\ldots\alpha_{2n}=0,\alpha_{2n+1}\ldots\alpha_{3n}=0\}$,
$B=\{(\alpha_1,\ldots,\alpha_{3n})|\alpha_1\ldots\alpha_n=0,\alpha_{n+1}\ldots\alpha_{2n}=1,\alpha_{2n+1}\ldots\alpha_{3n}=0\}$,
$C=\{(\alpha_1,\ldots,\alpha_{3n})|\alpha_1\ldots\alpha_n=0,\alpha_{n+1}\ldots\alpha_{2n}=0,\alpha_{2n+1}\ldots\alpha_{3n}=1\}$,
$D=\{(1,\ldots,1)\}$.

Assume we have a coset covering for the set of solutions of the equation

$$x_1x_2\ldots x_n+x_{n+1}x_{n+2}\ldots x_{2n}=0 \tag{5}$$

of the length $g(n)$. We can construct a coset covering for the Eq. (1) as follows. First we form a covering for the set A expanding all $2n$-dimensional vectors in cosets of the covering for the Eq. (5) to $3n$-dimensional by placing the n-dimensional vector $(1,1,\ldots,1)$ at the beginning of each $2n$-dimensional vector in cosets of the covering for the Eq. (5). Similarly we can cover B and C placing the vector $(1,1,\ldots,1)$ as appropriate in the middle and at the end of $2n$-dimensional vectors. Finally, D is covered by a 0-dimensional coset that consists of the $3n$-dimensional vector $(1,1,\ldots,1)$. Hence, we obtain a covering of the length equal to $3g(n)+1$.

Using Lemma, we conclude that a $(n-2)$-blocking set for $F_2^n\backslash\{(0,0,\ldots,0)\}$ with not more than $3n^{\log_2 3}$ elements from Theorem 2 can be converted into a covering by hyperplanes for $F_2^n\backslash\{(0,0,\ldots,0)\}$ of the same length, such that any pair of vectors in $F_2^n\backslash\{(0,0,\ldots,0)\}$ belongs to at least one of the hyperplanes from the covering. Then, using Proposition 5, we can obtain a covering of the same length by hyperplanes of the set $F_2^n\backslash\{(1,1,\ldots,1)\}$ such that any pair of vectors in $F_2^n\backslash\{(1,1,\ldots,1)\}$ belongs to at least one of the hyperplanes of the covering. Finally, using Proposition 3, we obtain a coset covering for the Eq. (5) of a length not greater than $3n^{\log_2 3}+1$. Hence, $g(n)\le 3n^{\log_2 3}+1$ and the covering for the Eq. (1) constructed above has a length not greater than $3g(n)+1\le 9n^{\log_2 3}+4$. □

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