

Mathematics

ON THE REMINDER OF A TAYLOR–MACLAURIN TYPE GENERAL FORMULA

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We continue the study started in [1] regarding the Taylor–Maclaurin representation of functions. In the present paper we obtain a more convenient representation for the function $R_n(x)$, the residual term of the generalized Taylor–Maclaurin formula.

Keywords: Weil operators, Taylor–Maclaurin type formulae.

§ 1. Preliminaries. It is known that the function

$$E_\rho(z; \mu) = \sum_{n=0}^{\infty} (z^n / \Gamma(\mu + n\rho^{-1})) (\rho > 0) \tag{1.1}$$

is an entire function of order ρ and type 1 for any value of the parameter μ [2].

For any $\mu > 0, \alpha > 0$ the following formula holds

$$1/\Gamma(\alpha) \int_0^z (z-t)^{\alpha-1} E_\rho(\lambda t^{1/\rho}; \mu) t^{\mu-1} dt = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho}; \mu + \alpha), \tag{1.2}$$

where λ is complex parameter, and the integration is taken along the line segment connecting points 0 and z (see [2], chap. III, (1.16)).

It can be shown also that for any $\alpha > 0, \beta > 0$ the following formula holds:

$$\int_0^1 x^{\alpha-1} E_\rho(\lambda x^{1/\rho}; \alpha) (1-x)^{\beta-1} E_\rho(\lambda^* (1-x)^{1/\rho}; \beta) dx = \\ = l^{\alpha+\beta-1} \lambda E_\rho(l^{1/\rho} \lambda; \alpha + \beta) - \lambda^* E_\rho(l^{1/\rho} \lambda^*; \alpha + \beta) / (\lambda - \lambda^*), \tag{1.3}$$

where λ, λ^* ($\lambda \neq \lambda^*$) are arbitrary complex parameters (see [2], chap. III, (1.21)).

From (1.3), using the well-known formula $E_\rho(z; 2/\rho) = 1/z \{E_\rho(z; 1/\rho) - 1/\Gamma(\rho^{-1})\}$,

for the particular case of $\alpha = \beta = 1/\rho$ we can deduce that

$$\int_0^x E_\rho(\lambda t^{1/\rho}; 1/\rho) t^{1/\rho-1} E_\rho(\lambda^* (x-t)^{1/\rho}; 1/\rho) (x-t)^{1/\rho-1} dt = \\ = x^{1/\rho-1} (E_\rho(\lambda x^{1/\rho}; 1/\rho) - E_\rho(\lambda^* x^{1/\rho}; 1/\rho)) / \lambda - \lambda^*. \tag{1.4}$$

The following classes of functions were introduced in [3]:

1) $C_\alpha^{(\infty)}$ ($\alpha \in [0,1)$) is the class of functions $f(x)$, possessing all derivatives

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$D_\infty^n f(x) \equiv f^{(n)}(x)$, $n = 0, 1, 2, \dots$, with $\sup_{0 \leq x < \infty} |(1+x^{m\alpha})f^{(n)}(x)| < +\infty$, $n, m = 0, 1, \dots$;

2) $C_\alpha^{+(\infty)}$ ($\alpha \in [0, 1)$) is the class of continuous functions $f(x)$ on $[0, +\infty)$, possessing all Weil derivatives $D_\infty^{n/\rho} f(x)$, $n = 0, 1, 2, \dots$, and satisfying $\sup_{0 \leq x < +\infty} |(1+x^{m\alpha})D_\infty^{n/\rho} f(x)| < +\infty$, $n, m = 0, 1, \dots$. It is proved that $C_\alpha^{(\infty)}$, $C_\alpha^{+(\infty)}$ coincide [3].

§ 2. Main Results.

Lemma 2.1. Let $\rho \geq 1$ ($1 - \alpha = 1/\rho$), $f(x) \in C_\alpha^{+(\infty)}$, $e^{\lambda^\rho x} |f(x)| \in L(0, +\infty)$.

Then the following formula holds

$$\int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) f(t_1) dt_1 = \\ = 1/(\lambda - \lambda_1) \int_x^{+\infty} f(t_1) \{e_\rho(t_1 - x; \lambda_1) - e_\rho(t_1 - x; \lambda_0)\} dt_1, \quad (2.1)$$

where $\lambda_0, \lambda_1 > 0$, $\lambda = \max\{\lambda_0, \lambda_1\}$, $e_\rho(x, \lambda) = E_\rho(\lambda x^{1/\rho}; 1/\rho) x^{1/\rho-1}$.

Proof. First note that the inner integral of the left hand side of (2.1) converges absolutely and uniformly on $t_0 \in (0, +\infty)$ due to Lemma 2.1 (see [1]).

Then the change the order of integration and formula (1.4) yield $\int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \times$
 $\times \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) f(t_1) dt_1 = \frac{1}{\lambda_1 - \lambda_0} \int_x^{+\infty} f(t_1) \{e_\rho(t_1 - x; \lambda_1) - e_\rho(t_1 - x; \lambda_0)\} dt_1.$

Lemma 2.2. Let $\{\lambda_j\}_0^\infty$ ($\lambda_i \neq \lambda_j$) be an arbitrary sequence of complex numbers.

Then for any $n \geq 1$.

$$\sum_{i=0}^n 1/\left(\prod_{j=0, (i \neq j)}^n (\lambda_j - \lambda_i)\right) = 0. \quad (2.2)$$

Lemma 2.3. Let $\rho \geq 1$ ($1 - \alpha = 1/\rho$), $f(x) \in C_\alpha^{+(\infty)}$ and assume that $e^{\lambda_i^\rho x} |f(x)| \in L(0, +\infty)$ for any $n \geq 1$. Then the following formula holds:

$$\int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) dt_1 \dots \int_{t_{n-1}}^{+\infty} e_\rho(t_n - t_{n-1}; \lambda_n) f(t_n) dt_n = \\ = \int_x^{+\infty} K_\rho(\tau - x; \{\lambda_i\}_0^n) f(\tau) d\tau, \quad x \in (0, +\infty), \quad (2.3)$$

where

$$K_\rho(x; \{\lambda_i\}_0^n) = \sum_{i=0}^n a_i^{(n)} e_\rho(x; \lambda_i), \quad (2.4)$$

$$a_i^{(n)} = \left\{ \prod_{j=0, (i \neq j)}^n (\lambda_j - \lambda_i) \right\}^{-1}, e_\rho(x; \lambda_i) = E_\rho(\lambda_i x^\rho; 1/\rho) x^{\frac{1}{\rho}-1}, 0 < \lambda_i < \lambda_{i+1} < +\infty, i = 0, 1, \dots$$

Proof. Let $n = 1$. Then, using (2.1), we get

$$\begin{aligned}
& \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) f(t_1) dt_1 = \\
& = 1/(\lambda_0 - \lambda_1) \int_x^{+\infty} f(t_1) \{e_\rho(t_1 - x; \lambda_0) - e_\rho(t_1 - x; \lambda_1)\} dt_1 = \\
& = \int_x^{+\infty} \{a_0^{(1)} e_\rho(t_1 - x; \lambda_0) + a_1^{(1)} e_\rho(t_1 - x; \lambda_1)\} f(t_1) dt_1 = \int_x^{+\infty} K_\rho(t_1 - x; \{\lambda_i\}_0^1) f(t_1) dt_1,
\end{aligned} \tag{2.5}$$

where $K_\rho(t_1 - x; \{\lambda_i\}_0^1) = \sum_{i=0}^1 a_i^{(1)} e_\rho(t_1 - x; \lambda_i)$, $a_i^{(1)} = \left\{ \prod_{j=0, (i \neq j)}^n (\lambda_i - \lambda_j) \right\}^{-1}$ ($i = 0, 1$).

So, the Lemma is true for $n = 1$.

Now we apply the method of mathematical induction. We suppose, the formula is true for n , and we prove that it remains true for $n + 1$. By our assumption,

$$\begin{aligned}
& \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) dt_1 \int_{t_1}^{+\infty} e_\rho(t_2 - t_1; \lambda_2) dt_2 \dots \int_{t_n}^{+\infty} e_\rho(t_{n+1} - t_n; \lambda_{n+1}) f(t_{n+1}) dt_{n+1} = \\
& = \int_{t_0}^{+\infty} K_\rho(t_1 - t_0; \{\lambda_i\}_1^{n+1}) f(t_1) dt_1,
\end{aligned} \tag{2.6}$$

where $K_\rho(t_1 - t_0; \{\lambda_i\}_1^{n+1}) = \sum_{i=1}^{n+1} \tilde{a}_i^{(n+1)} e_\rho(t_1 - t_0; \lambda_i)$, $\tilde{a}_i^{(n+1)} = \left\{ \prod_{j=1, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) \right\}^{-1}$. (2.7)

Now, using (2.6) and Lemma 2.1, we will have

$$\begin{aligned}
& \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} K_\rho(t_1 - t_0; \{\lambda_i\}_1^{n+1}) f(t_1) dt_1 = \\
& = \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} \left\{ \sum_{i=1}^{n+1} \tilde{a}_i^{(n+1)} e_\rho(t_1 - t_0; \lambda_i) \right\} f(t_1) dt_1 = \\
& = \int_x^{+\infty} f(t_1) \left\{ \sum_{i=1}^{n+1} \frac{\tilde{a}_i^{(n+1)}}{\lambda_0 - \lambda_i} e_\rho(t_1 - x; \lambda_0) + \sum_{i=1}^{n+1} \frac{\tilde{a}_i^{(n+1)}}{\lambda_i - \lambda_0} e_\rho(t_1 - x; \lambda_i) \right\} dt_1.
\end{aligned} \tag{2.8}$$

Next we show that

$$\sum_{i=1}^{n+1} \frac{\tilde{a}_i^{(n+1)}}{\lambda_0 - \lambda_i} = a_0^{(n+1)} = \left\{ \prod_{j=1}^{n+1} (\lambda_0 - \lambda_j) \right\}^{-1}, \tag{2.9}$$

$$\sum_{i=1}^{n+1} \tilde{a}_i^{(n+1)} / (\lambda_0 - \lambda_i) = \sum_{i=1}^{n+1} \left((\lambda_0 - \lambda_i) \prod_{j=1, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) \right)^{-1} = - \sum_{i=1}^{n+1} \left(\prod_{j=0, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) \right)^{-1} \tag{2.10}$$

(see (2.7)).

According to Lemma 2.2, $\sum_{i=0}^{n+1} 1 / \prod_{j=0, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) = 0$, hence,

$$- \sum_{i=1}^{n+1} \left(\prod_{j=0, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) \right)^{-1} = \left\{ \prod_{j=1}^{n+1} (\lambda_0 - \lambda_j) \right\}^{-1}. \tag{2.11}$$

Further we note that

$$\tilde{a}_i^{(n+1)} / (\lambda_i - \lambda_0) = 1 / \prod_{j=0, (i \neq j)}^{n+1} (\lambda_i - \lambda_j) = a_i^{(n+1)}, \quad (2.12)$$

and using (2.9)–(2.12), we get

$$\begin{aligned} & \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) dt_1 \dots \int_{t_n}^{+\infty} e_\rho(t_{n+1} - t_n; \lambda_{n+1}) f(t_{n+1}) dt_{n+1} = \\ & = \int_x^{+\infty} \left(\sum_{i=0}^{n+1} a_i^{(n+1)} e_\rho(t_0 - x; \lambda_i) \right) f(t_0) dt_0 = \int_x^{+\infty} K_\rho(t_0 - x; \{\lambda_i\}_0^{n+1}) f(t_0) dt_0. \end{aligned}$$

The Lemma is proved.

Theorem. Let $f(x) \in C_\alpha^{+(\infty)}$ and $e^{\lambda_n^\rho x} |D_\infty^{k/\rho} f(x)| \in L(0, +\infty)$, $k = 0, 1, \dots, n$ for any $n \geq 1$. Then the following formula holds:

$$f(x) = \sum_{i=0}^n L_\infty^{i/\rho} f(0) \varphi_i(x) + (-1)^{n+1} \int_x^{+\infty} K_\rho(\tau - x; \{\lambda_i\}_0^n) L_\infty^{(n+1)/\rho} f(\tau) d\tau, \quad (2.13)$$

where $K_\rho(x; \{\lambda_i\}_0^n) = \sum_{i=0}^n \left\{ \prod_{j=0, (i \neq j)}^n (\lambda_i - \lambda_j) \right\}^{-1} e_\rho(x; \lambda_i)$, $e_\rho(x; \lambda_i) = E_\rho(\lambda_i x^\rho; \frac{1}{\rho}) x^{\frac{1}{\rho}-1}$.

Proof. According to Theorem 3.3 (see [1]),

$$f(x) = \sum_{i=0}^n L_\infty^{i/\rho} f(0) \varphi_i(x) + R_n(x), \quad x \in (0, +\infty), \quad (2.14)$$

where $R_n(x) = (-1)^{n+1} \int_x^{+\infty} e_\rho(t_0 - x; \lambda_0) dt_0 \int_{t_0}^{+\infty} e_\rho(t_1 - t_0; \lambda_1) dt_1 \dots \int_{t_{n-1}}^{+\infty} e_\rho(t_n - t_{n-1}; \lambda_n) L_\infty^{\frac{n+1}{\rho}} f(t_n) dt_n$.

Since $e^{\lambda_n^\rho x} |D_\infty^{k/\rho} f(x)| \in L(0, +\infty)$ ($k = 0, 1, 2, \dots, n$) for any $n \geq 1$ and $L_\infty^{(n+1)/\rho} f(x) = \prod_{k=0}^n (D_\infty^{k/\rho} + \lambda_k) f(x)$. Then it is easy to see that $e^{\lambda_n^\rho x} |L_\infty^{(n+1)/\rho} f(x)| \in L(0, +\infty)$.

Now, using Lemma 2.3, we get

$$R_n(x) = (-1)^{n+1} \int_x^{+\infty} K_\rho(\tau - x; \{\lambda_i\}_0^n) L_\infty^{(n+1)/\rho} f(\tau) d\tau. \quad (2.15)$$

From (2.14) and (2.15) we obtain (2.13).

The Theorem is proved.

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**Թեյլոր–Մակլորենի մի ընդհանրացված բանաձևի
մնացորդային անդամի մասին**

Աշխատանքում շարունակվում է [1]-ում սկսված Թեյլոր–Մակլորենի դասի ֆունկցիաների հետազոտությունը: Ստացվել է $R_n(x)$ մնացորդային անդամի համար ավելի հարմար բանաձև:

**Об остаточном члене одной обобщенной формуле
типа Тейлора–Маклорена**

В работе продолжается изучение обобщенной формулы Тейлора–Маклорена для функций некоторых классов. Получена более удобная формула для остаточного члена.