

COMMUNICATIONS

*Informatics*

Approximation by poised sets of nodes

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In the present paper it has been shown that nodes of any finite set  $X \subset \mathbb{R}^d$  can be made independent by arbitrarily small perturbation, in other words, the set  $X$  can be approximated by sets of independent nodes. In the case of  $\#X = \dim \Pi_n^d$  the set  $X$  can be approximated by sets of poised nodes.

**Keywords:** Lagrange interpolation, independent points, poised sets.

**Introduction.** Let  $\Pi_n^d$  be the space of all polynomials in  $d$  variables of total degree  $\leq n$ .

*Definition 1.* The interpolation problem with a set of nodes  $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$  and  $\Pi_n^d$  is called poised (unisolvent), if for any data  $\{c_1, \dots, c_s\}$  there is a unique polynomial  $p \in \Pi_n^d$  such that  $p(\mathbf{x}_i) = c_i, i = 1, \dots, s$ .

This gives us a system of  $s$  linear equations with  $\binom{n+d}{d}$  unknowns – the coefficients of polynomial. The poisedness means that this system has a unique solution for any right-hand side values. This implies the following necessary condition of poisedness:  $s = N := N_n^d = \dim \Pi_n^d = \binom{n+d}{d}$ .

Thus, we consider poisedness, only if this condition holds. In the latter case we have: the linear system is unisolvent for arbitrary right-hand side values  $c_i, i = 1, 2, \dots, N$ , iff the corresponding homogeneous system has no solution except the trivial one. So, we can state the following

*Proposition 1.* The interpolation problem with a set of nodes  $X = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, N\}$  and  $\Pi_n^d$  is poised, iff  $p \in \Pi_n^d, p(\mathbf{x}_i) = 0, i = 1, \dots, N \Rightarrow p = 0$ .

Consider a set of nodes  $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$ . A polynomial  $p$  is called a fundamental polynomial for  $A := \mathbf{x}_k \in X_s$  and  $\Pi_n^d$ , if  $p \in \Pi_n^d, p(\mathbf{x}_k) = 1, p(\mathbf{x}_i) = 0, 1 \leq i \leq s, i \neq k$ .

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We denote fundamental polynomial of  $A := \mathbf{x}_k \in X_s$  by  $p_k^* := P_{\{A, X_s\}}^*$ .

*Definition 2.* The set of nodes  $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$  is called  $\Pi_n^d$ -independent, if all fundamental polynomials  $p_i^* \in \Pi_n^d$ ,  $i = 1, \dots, s$ , exist. Otherwise,  $X_s$  is called dependent for  $\Pi_n^d$ , i.e.  $X_s$  is dependent for  $\Pi_n^d$ , iff

$$p \in \Pi_n^d, \quad p(\mathbf{x}_i) = 0, \quad 1 \leq i \leq s, \quad i \neq i_0 \Rightarrow p(\mathbf{x}_{i_0}) = 0.$$

The following statements are evident.

*Proposition 2.* Assume that a set of nodes  $X$  is  $\Pi_n^d$ -independent. Then:

1. Any nonempty subset  $Y \subset X$  is  $\Pi_n^d$ -independent.
2.  $\#X \leq N = \dim \Pi_n^d$ , and if  $\#X = N$ , then set  $X$  is  $\Pi_n^d$ -poised.

In the sequel we use the following notation. Let us enumerate the monomials in  $d$  variables and denote them by  $\varphi_j(\mathbf{x})$ ,  $\mathbf{x} = (x_1, \dots, x_d)$ ,  $j = 1, 2, \dots$ . Numbering is performed such that monomials of the same degree are numbered in arbitrary order, e.g. lexicographical, and the monomials of different degrees are numbered in increasing order. In particular,  $\varphi_1(\mathbf{x}) = 1$ . For example, if the monomials of the same degree are numbered in lexicographical order, then

$$\varphi_2(\mathbf{x}) = x_1, \dots, \quad \varphi_{d+1}(\mathbf{x}) = x_d, \quad \varphi_{d+2}(\mathbf{x}) = x_1^2, \quad \varphi_{d+3}(\mathbf{x}) = x_1 x_2, \dots$$

Thus, all monomials in  $d$  variables of degree not exceeding  $n$  are  $\varphi_j(\mathbf{x})$ ,  $j = 1, 2, \dots, N$ .

Now let  $X = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$  be a set of nodes with  $s \leq N$ . The Vandermonde matrix  $V_n(X) = [\varphi_1(\mathbf{x}_i), \varphi_2(\mathbf{x}_i), \dots, \varphi_N(\mathbf{x}_i)]_{i=1}^s$  is an  $s \times N$ -matrix with  $s = \#X$  rows and  $N = \dim \Pi_n^d = \binom{n+d}{d}$  columns. It is easy to see that  $V_n(X)$  is of full rank, i.e.  $\text{rank } V_n(X) = \#X$ , iff the points of  $X$  are  $\Pi_n^d$ -independent.

**Approximation by Poised Sets of Nodes.** In the sequel we use the following [1]

**Lemma.** Let  $A = [a_{i1}, a_{i2}, \dots, a_{is}]_{i=1}^r$ ,  $r \geq 1$  be a matrix of rank  $r$  and  $\Omega \subset \mathbb{R}^d$  be a set such that for any  $k = 1, 2, \dots$  it contains points not lying on hyper-surface of order  $k$ . Then, there exists a point  $\mathbf{x}^{(1)} \in \Omega$  such that the rank of matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rs} \\ \varphi_1(\mathbf{x}^{(1)}) & \varphi_2(\mathbf{x}^{(1)}) & \cdots & \varphi_s(\mathbf{x}^{(1)}) \end{pmatrix} \text{ is } r+1.$$

Let  $\varepsilon > 0$ ,  $\mathbf{x}_0 \in \mathbb{R}^d$  and  $B(\mathbf{x}_0, \varepsilon)$  be the open ball with center  $\mathbf{x}_0$  and radius  $\varepsilon$ :  $B(\mathbf{x}_0, \varepsilon) := \{\mathbf{x}_0 \in \mathbb{R}^d : \|\mathbf{x}_0 - \mathbf{x}\| < \varepsilon\}$ .

From Lemma we get the following

**Proposition 3.** Suppose  $X = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$  is  $\Pi_n^d$ -independent set of points. If  $s < N$ , then for any  $\varepsilon > 0$  and  $\mathbf{x}_0 \in \mathbb{R}^d$  there exists a point  $\mathbf{x}' \in B(\mathbf{x}_0, \varepsilon)$  such that the set  $X \cup \{\mathbf{x}'\}$  is  $\Pi_n^d$ -independent.

*Proof.* Consider the Vandermonde matrix  $V_n(X)$ . From  $\Pi_n^d$ -independence of  $X$  we have that  $\text{rank} V_n(X) = \#X = s$ . Since any hypersurface cannot contain a point with its neighbourhood in  $\mathbb{R}^d$ , then the condition of Lemma holds for  $\Omega = B(\mathbf{x}_0, \varepsilon)$ . Thus, there exists a point  $\mathbf{x}' \in \Omega$  such that  $\text{rank} V_n(X \cup \{\mathbf{x}'\}) = s + 1$ . So, the set  $X \cup \{\mathbf{x}'\}$  is  $\Pi_n^d$ -independent.

The following theorem shows that any finite set of nodes  $X \subset \mathbb{R}^d$  can be approximated by a set of independent nodes.

**Theorem.** Let  $X_s = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$ ,  $s \leq N$ . For arbitrary  $\varepsilon > 0$  there exists a  $\Pi_n^d$ -independent set  $Y_s = \{\mathbf{y}_i \in \mathbb{R}^d : i = 1, 2, \dots, s\}$  such that  $\|\mathbf{x}_i - \mathbf{y}_i\| < \varepsilon$ ,  $i = 1, 2, \dots, s$ .

*Proof.* We shall prove the statement using the method of mathematical induction on  $s$ . Clearly, the statement holds for  $s = 1$ . Suppose the statement holds for all  $k$ ,  $k < s$ . Consider the set  $X_{s-1} = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, s-1\}$ . The induction hypothesis implies that for arbitrary  $\varepsilon > 0$  there exists a  $\Pi_n^d$ -independent set  $Y_{s-1} = \{\mathbf{y}_i \in \mathbb{R}^d : i = 1, 2, \dots, s-1\}$  such that  $\|\mathbf{x}_i - \mathbf{y}_i\| < \varepsilon$ ,  $i = 1, 2, \dots, s-1$ . By the Proposition 3 there exists a point  $\mathbf{y}_s \in B(\mathbf{x}_s, \varepsilon)$  such that the set  $Y_s = Y_{s-1} \cup \{\mathbf{y}_s\}$  is  $\Pi_n^d$ -independent.

Now we readily get the following

**Corollary.** Consider a set of points  $X = \{\mathbf{x}_i \in \mathbb{R}^d : i = 1, 2, \dots, N\}$ . For arbitrary  $\varepsilon > 0$  there exists a  $\Pi_n^d$ -poised set  $Y = \{\mathbf{y}_i \in \mathbb{R}^d : i = 1, 2, \dots, N\}$  such that  $\|\mathbf{x}_i - \mathbf{y}_i\| < \varepsilon$ ,  $i = 1, 2, \dots, N$ .

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բազմություններով էջ. 60–62

Ներկա աշխատանքում ցույց է տրված, որ ցանկացած  $X \subset \mathbb{R}^d$  վերջավոր  
բազմության ցանկացած հանգույցներ, որքան ասես փոքր չափով  
տեղաշարժելով, այն կարելի է դարձնել անկախ հանգույցների բազմու-  
թյուն, կամ այլ կերպ ասած, բազմությունը կարելի է մոտարկել անկախ  
հանգույցների բազմություններով: Երբ  $\#X = \dim \Pi_n^d$ ,  $X$  բազմությունը կարելի  
է մոտարկել հանգույցների ճշգրիտ բազմություններով:

**Г. С. Авагян, Л. Р. Рафаелян. Приближение множествами корректных узлов**  
стр. 60–62

В работе доказывается, что узлы любого конечного множества  $X \subset \mathbb{R}^d$   
путем перемещения в сколь угодно малой окрестности можно сделать  
независимыми или, другими словами, множество  $X$  может быть прибли-  
жено множествами независимых узлов. В случае  $\#X = \dim \Pi_n^d$  множество  $X$   
может быть приближено множествами корректных узлов.