PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2017, **51**(1), p. 13–17

Mathematics

ON A SUMMATION METHOD FOR VILENKIN AND GENERALIZED HAAR SYSTEMS

G. G. GEVORKYAN [1] *, K. A. NAVASARDYAN [2]**

[1] *Chair of Mathematical Analysis and Functions Theory YSU, Armenia*
[2] *Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia*

New summation method for Vilenkin and Haar series is considered. In particular, it is shown that the Fourier series of an integrable function is almost everywhere summable by considered method.

MSC2010: 42C10.

***Keywords*:** Vilenkin system, generalized Haar system, summation method.

Introduction. The Fourier series of an integrable function with respect to Vilenkin system can be divergent almost everywhere (see [1, 2]). The same holds also for Fourier series with respect to generalized Haar system generated by an unbounded sequence p_k. In the present paper we introduce a new summation method for series with respect to Vilenkin and generalized Haar systems and prove some properties of this method. In particular, we prove the almost everywhere (a.e.) convergence of sums of Fourier series of an integrable function by that method.

We recall the definitions of the Vilenkin system (see [3]). Let $p_k,\ p_k \geq 2,\ k \in \mathbb{N}$, be a sequence of natural numbers and $m_0 = 1,\ m_{k+1} = m_k p_{k+1}$. Then every nonnegative integer n is uniquely represented by the series

$$n = \sum_{k=1}^{\infty} n_k m_{k-1}, \text{ where } n_k \in \{0,1,...,p_k-1\}, \ k \in \mathbb{N}.$$

Every point $x \in [0,1)$ can be represented in the following way:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{m_k}, \text{ where } x_k \in \{0,1,...,p_k-1\}, \ k \in \mathbb{N}.$$

If there are two different representations, we choose the one for which $\lim\limits_{k\to\infty} x_k = 0$. The functions

$$R_k(x) := \exp\left(\frac{2\pi i x_k}{p_k}\right), \ k \in \mathbb{N},$$

are called generalized Rademacher functions.

* E-mail: ggg@ysu.am **E-mail: knavasard@ysu.am

The Vilenkin system is given by

$$\psi_0(x) \equiv 1 \text{ and } \psi_n(x) := \prod_{k=1}^{\infty} R_k^{n_k}(x) \text{ for } n \in \mathbb{N}. \tag{1}$$

Note, that in the case $p_k = 2$, $k \in \mathbb{N}$, the Vilenkin system coincides with the Walsh system.

For $n = m_k + r(p_{k+1} - 1) + s - 1$, where $0 \le r \le m_k - 1$, $1 \le s \le p_{k+1} - 1$, we put (see [3])

$$\chi_n(x) := \chi_{r,s}^k(x) := \begin{cases} \sqrt{m_k} \exp\left(2\pi i \dfrac{x_{k+1}}{p_{k+1}} s\right) & \text{when } x \in \left[\dfrac{r}{m_k}, \dfrac{r+1}{m_k}\right), \\ 0 & \text{when } x \notin \left[\dfrac{r}{m_k}, \dfrac{r+1}{m_k}\right). \end{cases} \tag{2}$$

The system $\left\{\chi_n(x)\right\}_{n=0}^{\infty}$, where $\chi_0(x) \equiv 1$, is called the generalized Haar system generated by sequence p_k, $k \in \mathbb{N}$. When $p_k = 2$, $k \in \mathbb{N}$, the generalized Haar system coincides with the classical Haar system.

The Vilenkin and the generalized Haar systems are well investigated in the mathematical literature. When $\sup_k p_k < \infty$, the Vilenkin and the generalized Haar systems properties are very similar with the ones of Walsh and Haar systems, correspondingly.

Definition of the Summation Method. In this section we define a new method of summation for Vilenkin and generalized Haar systems. We denote

$$\mathfrak{I}_k = \left\{ \left[\frac{j}{m_{k+1}}, \frac{j+1}{m_{k+1}}\right) : \ j = 0, 1, ..., m_{k+1} - 1 \right\}, \ k = 1, 2, ...$$

For $J \in \mathfrak{I}_k$, we denote by $\widetilde{J}$ the interval from $\mathfrak{I}_{k-1}$, containing J. The intervals J_l, $l \in \{0, \pm 1, \pm 2, ...,\}$, are defined in the following way:

1. $J_l \subset \widetilde{J}$, $J_0 = J$.
2. The right endpoint of J_l coincides with the left endpoint of J_{l+1}, with the convention that we identify the endpoints of $\widetilde{J}$, in particular, if the right endpoint of J_l is $\dfrac{j}{m_k}$, then the left endpoint of J_{l+1} is $\dfrac{j-1}{m_k}$.

We put $J^q = \bigcup_{l=-q}^{q} J_l$, $q = 0, 1, 2, ...$ It is clear that $J^0 = J$.

For $x \in [0,1)$ and $k \in \mathbb{N}$ we denote by $I_{k,x}$ the interval satisfying $I_{k,x} \in \mathfrak{I}_k$ and $x \in I_{k,x}$. For $q \in \left\{1, 2, ..., \left[\dfrac{p_{k+1}}{2}\right]\right\}$ we denote

$$\varphi_{k,x}^q(t) = \begin{cases} \dfrac{m_{k+1}}{q}\left(1 - \dfrac{|l|}{q}\right), & \text{if } t \in (I_{k,x})_l \text{ and } |l| < q, \\ 0, & \text{if } t \notin I_{k,x}^{q-1}. \end{cases} \tag{3}$$

Let in the sequel $\{f_n(x)\}_{n=0}^{\infty}$ be one of the systems (1), (2). We will consider series

$$\sum_{n=0}^{\infty} a_n f_n(x). \tag{4}$$

Taking into account the definitions of the system $\left\{f_n(x)\right\}_{n=0}^{\infty}$, it is clear that for each $\varphi_{k,x}^q$, we have

$$(f_n, \varphi_{k,x}^q) := \int_0^1 f_n(t) \varphi_{k,x}^q(t) dt = 0 \text{ when } n \ge m_{k+1}.$$

Therefore, for each series (4) and $x \in [0,1)$, every k and q the sums

$$S_{k,q}(x) := \sum_{n=0}^{\infty} a_n \int_0^1 f_n(t)\varphi_{k,x}^q(t)dt \tag{5}$$

are well-defined. We denote

$$S^*(x) := \sup_{k,q} |S_{k,q}(x)|. \tag{6}$$

It is not difficult to check that (5) defines a linear method of summation, i.e. there exist numbers $\alpha_n^{k,q}$ and $\beta_n^{k,q}$, such that:

1. $S_{k,q}(x) = \sum_{n=0}^{m_{k+1}-1} \alpha_n^{k,q} a_n \psi_n(x)$ when $\{f_n\}_{n=0}^{\infty} = \left\{\psi_n\right\}_{n=0}^{\infty}$;
2. $S_{k,q}(x) = \sum_{n=0}^{m_{k+1}-1} \beta_n^{k,q} a_n \chi_n(x)$ when $\{f_n\}_{n=0}^{\infty} = \left\{\chi_n\right\}_{n=0}^{\infty}$.

Note that this summation method differs from methods considered in the literature for systems $\{f_n\}_{n=0}^{\infty}$. For comparison see [4].

Theorems. It is clear, that if the series (4) is the Fourier series of an integrable function f, then

$$S_{k,q}(f,x) := S_{k,q}(x) = \int_0^1 f(t)\varphi_{k,x}^q(t)dt \tag{7}$$

and

$$S^*(f,x) := S^*(x) \le \sup_{\substack{k,q:\\ 1\le q\le [\frac{p_{k+1}}{2}]}} \int_{I_{k,x}^q} |f(t)\varphi_{k,x}^q(t)|dt. \tag{8}$$

Denote

$$\mathcal{M}^*(f,x) = \sup_{\substack{k,q:\\ 0\le q\le [\frac{p_{k+1}}{2}]-1}} \frac{1}{|I_{k,x}^q|}\int_{I_{k,x}^q} |f(t)|dt.$$

Lemma. The inequality $S^*(f,x) \le \mathcal{M}^*(f,x)$ holds for each $f \in L_1$.

Proof. Indeed, for each $I_{k,x}^q$, $\left(1 \le q \le \left[\frac{p_{k+1}}{2}\right]\right)$ using (3) we will get

$$\int_{I_{k,x}^q} |f(t)\varphi_{k,x}^q(t)|dt = \frac{m_{k+1}}{q^2}\sum_{\nu=0}^{q-1}\int_{I_{k,x}^\nu} |f(t)|dt \le$$

$$\mathcal{M}^*(f,x)\frac{m_{k+1}}{q^2}\sum_{\nu=0}^{q-1}\frac{2\nu+1}{m_{k+1}} = \mathcal{M}^*(f,x). \qquad \square$$

Theorem 1. For any integrable function f and $\lambda > 0$

$$\operatorname{mes}\{x\in[0,1) : S^*(f,x) > \lambda\} \le \frac{3}{\lambda}\int_0^1 |f(t)|dt. \tag{9}$$

Proof. First, we prove that

$$\operatorname{mes}\{x\in[0,1) : \mathcal{M}^*(f,x) > \lambda\} \le \frac{3}{\lambda}\int_0^1 |f(t)|dt. \tag{10}$$

Let $E_\lambda = \{x \in [0,1) : \mathcal{M}^*(f,x) > \lambda\}$. Then for every $x \in E_\lambda$ there exists $I^q_{k,x}$ with $q \le \left[\frac{p_{k+1}}{2}\right] - 1$ such that

$$\frac{1}{|I^q_{k,x}|}\int_{I^q_{k,x}} |f(t)|dt > \lambda. \tag{11}$$

We denote the collection of all $I^q_{k,x}$, satisfying (11), by G. Then $E_\lambda = \bigcup_{I\in G} I$. Let $I_1 \in G$ be an interval from G with the biggest measure. Note that such I_1 (maybe not unique) exists, because $\operatorname{mes}(I^q_{k,x}) = \dfrac{2q+1}{m_{k+1}}$ and $0 \le q \le \left[\frac{p_{k+1}}{2}\right] - 1$. By induction we choose the sequence $I_j \in G$ with properties

$$G_j := \{I \in G : I \cap I_\nu = \emptyset,\ \ \nu = 1,2,...,j-1\} \tag{12}$$

and

$$\operatorname{mes}(I_j) = \max\{\operatorname{mes}(I) :\ I \in G_j\}. \tag{13}$$

Let $I \in G$ and j be the smallest natural number for which $I \cap I_j \ne \emptyset$. Of course, $I_j = I^{q_j}_{k_j,x_j}$ for some $k_j,\ x_j,\ q_j$. Let $\widehat{I_j} = I^{3q_j}_{k_j,x_j}$. It is not difficult to check that from (12), (13) it follows that $I \subset \widehat{I_j}$. Therefore, $E_\lambda \subset \bigcup_j \widehat{I_j}$. Hence, taking into account (11) and (12), we obtain

$$\operatorname{mes}(E_\lambda) \le 3\sum_j |I_j| < \frac{3}{\lambda}\sum_j \int_{I_j} |f(t)|dt = \frac{3}{\lambda}\int_{\bigcup I_j} |f(t)|dt \le \frac{3}{\lambda}\|f\|_1. \qquad \square$$

The next theorem follows from Theorem 1 by standard methods (see [5]).

Theorem 2. For any integrable function f,

$$\operatorname{mes}\{x \in [0,1) : S^*(f,x) > \lambda\} = o\left(\frac{1}{\lambda}\right). \tag{14}$$

Proof. Let $f_{1,\lambda}(t) = f(t)$, if $|f(t)| \le \lambda/2$ and $f_{1,\lambda}(t) = 0$, if $|f(t)| > \lambda/2$. Put $f_{2,\lambda} = f - f_{1,\lambda}$. Then, taking into account that $\|\varphi^q_{k,x}\|_1 = 1$, we obtain

$$\operatorname{mes}\{x \in [0,1) : S^*(f,x) > \lambda\} \le \operatorname{mes}\left\{x \in [0,1) : S^*(f_{1,\lambda},x) > \frac{\lambda}{2}\right\} +$$
$$\operatorname{mes}\left\{x \in [0,1) : S^*(f_{2,\lambda},x) > \frac{\lambda}{2}\right\} = \operatorname{mes}\left\{x \in [0,1) : S^*(f_{2,\lambda},x) > \frac{\lambda}{2}\right\} \le \frac{6}{\lambda}\|f_{2,\lambda}\|_1.$$

To complete the prove note that $\|f_{2,\lambda}\|_1 \to 0$ when $\lambda \to \infty$. $\square$

It is clear, that if f is continuous on $[0,1)$, then $\lim_{k\to\infty} S_{k,q}(f,x) = f(x)$ uniformly on $[0,1)$. Using this fact, one can obtain the following theorem.

Theorem 3. If $f \in L^1[0,1)$, then

$$\lim_{k\to\infty} S_{k,q}(f,x) = f(x) \ \text{ a.e. } \ \text{on } [0,1). \tag{15}$$

Proof. It is enough to prove that for each $z > 0$ the measure of the set

$$\mathcal{P}_z := \{x \in [0,1) : \limsup_{k\to\infty} |S_{k,q}(x) - f(x)| > z\}$$

is zero. For arbitrary $\varepsilon > 0$ we choose a continuous function g with $\|f-g\|_1 < \varepsilon$. Because $\lim_{k\to\infty} S_{k,q}(g,x) = g(x)$ uniformly, then

$$\operatorname{mes}(\mathcal{P}_z) = \operatorname{mes}\{x \in [0,1) : \limsup_{k\to\infty} |S_{k,q}(f-g,x) - (f(x)-g(x))| > z\} \le$$
$$\operatorname{mes}\left\{x : \limsup_{k\to\infty} |S_{k,q}(f-g,x)| > \frac{z}{2}\right\} + \operatorname{mes}\left\{x : |f(x)-g(x)| > \frac{z}{2}\right\} \le$$
$$\frac{6}{z}\|f-g\|_1 + \frac{2}{z}\|f-g\|_1 < \frac{8\varepsilon}{z}.$$

Then the proof of the Theorem follows, because ε is an arbitrary positive number. $\square$

We will use the properties of the considered summation method and the majorant $S^*(f,x)$ in our future investigations of uniqueness questions for series with respect to Vilenkin and generalized Haar systems.

This work was supported by State Committee of Science MES RA, in frame of research project SCS RA 10–3/1–41.

Received 18.11.2016

R E F E R E N C E S

1. **Kheladze Sh.V.** On Everywhere Divergence of Fourier–Walsh Series. // Soobshch. AN Gruz. SSR, 1975, v. 77, № 2, p. 305–307 (in Russian).
2. **Kheladze Sh.V.** On Everywhere Divergence of Fourier Series with Respect to a Bounded Vilenkin System. // Trudy Tbilisskogo Math. Inst. AN Gruz. SSR, 1978, v. 58, p. 225–242 (in Russian).
3. **Vilenkin N.J.** On a Class of Complete Orthonormal System. // Izv. Akad. Nauk SSSR. Matematika, 1947, v. 11, № 4, p. 363–400 (in Russian).
4. **Weisz F.** Summation of Fourier Series. // Acta Math. Acad. Paedagog. Nyhazi., 2004, v. 20, p. 239–266.
5. **De Guzman M.** Differentation of Integrals in R^n. M.: Mir, 1978, 200 p. (in Russian).