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ON ONE COMPLETE AND MINIMAL SET OF BUILT-IN CONSTANTS FOR BACKUS FP SYSTEM

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In the present paper the completeness and the minimality of the set of the following built-in constants of Backus FP system are proved: *Identity*, *Head, Tail, Append left, Equals, Composition, Constuction, Condition, Constant.*

***Keywords*:** Backus FP system, functional programming language, built-in constants, Turing completeness, minimality.

1. Introduction. In this paper we will consider the following subset of the set of built-in constants of Backus FP system [1]: the functions *Identity* (in short, *id*), *Head* (*hd*), *Tail* (*tl*), *Append left* (*apndl*), *Equals* (*eq*) and the functionals *Composition* (*comp*), *Construction* (*constr*), *Condition* (*cond*), *Constant* (*const*).We will show completeness and minimality of this subset. Let $\Phi=\{id, hd, tl, apndl, eq, comp, constr, cond, const\}$. Theorem 3.1 of the present paper states that the set of built-in constants Φ is complete. Theorem 3.2 states that the set of built-in constants Φ is minimal for Backus FP system, which uses more than three atoms. Theorem 3.3 asserts that the set of built-in constants $\Phi\backslash\{const\}$ is complete and minimal for Backus FP system, which uses only three atoms, and it is the only proper subset of the set Φ, which is complete for such systems.

2. Definitions Used and Preliminary Results. In the paper we will use the definitions given in [2]. Let $Atoms=\{a_1,\dots,a_n\}$, $n\geq 3$, be a finite set of atoms, containing at least three elements (T, F, $nil\in Atoms$). T and F correspond to logical true and false values respectively, nil corresponds to empty list.

Backus FP system is a functional programming language, which is defined with the following quadruple $L=(M, C, X, \Lambda)$, where:

• M is the set of objects: 1) if $m\in Atoms$, then $m\in M$; 2) $\perp\in M$; 3) if $m_1,\dots,m_n\in M$, $1\leq i\leq n$, $n\geq 0$, then $(m_1\dots m_n)\in M$, and if for any $1\leq i\leq n$, $m_i=\perp$, then $(m_1\dots m_n)=\perp$.

• $C=M\cup C_1\cup C_2$ is the set of constants, where C_1 and C_2 are correspondingly subsets of built-in functions and functionals of Backus FP system. We will consider the set of built-in functions $C_1=\{id, hd, tl, apndl, eq\}$ and the set of built-in functionals $C_2=\{comp, constr, cond, const\}$. Let us give their definitions ($m\in M$, $S=M\setminus\{\perp\}$ is the set of *S-expression*).

$id(m)=m$; $hd(m)=m_1$, if $m=(m_1\dots m_k)$, $m_i\in S$, $i=1,\dots k$, $k\geq 1$, $\perp$ otherwise;

$$tl(m)=\begin{cases} nil, \text{ if } m=(m_1), m_1\in S, \\ (m_2\dots m_k), \text{ if } m=(m_1\dots m_k),\ m_i\in S,\ i=1,\dots,k,\ k>1, \\ \perp, \text{ otherwise,}\end{cases}$$

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$$apndl(m)=\begin{cases}(m_0\, m_1 \ldots m_k), \text{ if } m=(m_0(m_1\ldots m_k)),\\ m_i\in S, i=1,\ldots,k,\ k>1,\\ (m_0), \text{ if } m=(m_0\, nil),\ m_0\in S,\\ \perp, \text{ otherwise,}\end{cases}\qquad eq(m)=\begin{cases}T, \text{ if } m=(m_1\, m_2),\ m_1=m_2,\ m_1,m_2\in S,\\ F, \text{ if } m=(m_1\, m_2),\ m_1\neq m_2,\ m_1,m_2\in S,\\ \perp, \text{ otherwise.}\end{cases}$$

comp$\in[[M\to M]^2\to[M\to M]]$ and for any functions $g,h\in[M\to M]$, $comp(g,h)=$ $=f\in[M\to M]$ is defined by: $f(m)=g(h(m))$ for all $m\in M$;

constr$\in[[M\to M]^2\to[M\to M]]$ and for any functions $g,h\in[M\to M]$, $constr(g,h)=$ $=f\in[M\to M]$ is defined by: $f(m)=(g(m)\ h(m))$ for all $m\in M$;

cond$\in[[M\to M]^3\to[M\to M]]$ and for any functions $p,g,h\in[M\to M]$, $cond(p,g,h)=$

$=f\in[M\to M]$ is defined by: $f(m)=\begin{cases}g(m), \text{ if } p(m)=T,\\ h(m), \text{ if } p(m)=F,\\ \perp, \text{ otherwise,}\end{cases}$ for all $m\in M$.

const$\in[M\to[M\to M]]$ and for any $m_0\in M$, $const(m_0)=f\in[M\to M]$ is defined by:

$$f(m)=\begin{cases}m_0, \text{ if } m\neq\perp,\\ \perp, \text{ otherwise,}\end{cases}\quad \text{for all } m\in M.$$

- X is the set of variables: $X=\{F_i \mid F_i\in V_{[M\to M]},\ i\geq 1\}$.
- Λ is the set of terms constructed using constants and variables only from the sets C and X, without using the 4th point of the definiton of term [2], and the elements of the set M are used only as the values of the argument of the functional *const.*

Definition 2.1. A program P in Backus FP system is a system of equations of the form

$$\begin{cases}F_1=\tau_1,\\ \cdots\\ F_n=\tau_n,\end{cases}\tag{1}$$

where $F_i\in X$, $i\neq j\Rightarrow F_i\neq F_j$, $\tau_i\in\Lambda\cap\Lambda_{[M\to M]}$, $FV(\tau_i)\subset\{F_1,\ldots,F_n\}$, $i,j=1,\ldots,n$, $n\geq 1$. The mapping $\Psi_P:[M\to M]^n\to[M\to M]^n$, where $\Psi_P(\bar{g})=<Val_{\bar{g}}(\tau_1),\ldots,\ Val_{\bar{g}}(\tau_n)>$, $\bar{g}\in[M\to M]^n$ will be called a mapping corresponding the program (1). By $(\bar{g})_i$, $i=1,\ldots,n$, we will denote ith component of the vector $\bar{g}$. We will say that $\bar{f}\in[M\to M]^n$ is the solution of the program (1), if $\Psi_P(\bar{f})=\bar{f}$. From [3] it follows that any program (1) has a least solution. We will devote $\Psi_P^0(\bar{\Omega})=\bar{\Omega}, \Psi_P^{j+1}(\bar{\Omega})=\Psi_P(\Psi_P^j(\bar{\Omega})), j\geq 0$, where $\bar{\Omega}$ is the least element of $[M\to M]^n$. Using the results from [4] we can deduce, that if $\bar{f}=<f_1,\ldots,f_n>$ is the least solution of the program P, then $f_i=\sup\{(\Psi_P^k(\bar{\Omega}))_i \mid k\geq 0\}$, because all constants used in the program P are continuous mappings and all variables are of order 1. The equation $F_1=\tau_1$ is called the main equation and the function $f_P=(\bar{f})_1$ – the fixpoint semantics of the program P.

Definition 2.2. We will say that the function $f\in[M\to M]$ is representable in Backus FP system, if there exists a program P such that $f_P=f$, where f_P is the fixpoint semantics of the program P.

Definition 2.3. The set of built-in constants Ψ, where $\Psi\subset(C_1\cup C_2)$, is called complete, if for any computable function f: $B\to S$, where $B\subset S$, there exists a program P of Backus FP system, which uses the set of constants $M\cup\Psi$ such that the following holds:

- if $m\in B$, then $f_P(m)=f(m)$;
- if $m\notin B$, then $f_P(m)=\perp$,

where f_P is the fixpoint semantics of the program P.

Definition 2.4. The complete set of built-in constants Ψ, where $\Psi\subset(C_1\cup C_2)$, is called minimal, if for any $\varphi\in\Psi$ the set of built-in constants $\Psi\backslash\{\varphi\}$ is not complete.

3. The Main Results.

Theorem 3.1. The set of built-in constants $C_1\cup C_2$ is complete.

The proof of the Theorem 3.1 is similar to the proof of the Main Theorem of the paper [5]. The alphabet of Turing machine is the set $U = Atoms\cup\{[,],–\}$, where "–" corresponds to blank symbol. Let Q be the finite set of internal states of Turing machine. Any $s\in S$ will be represented on the tape of a Turing machine as follows:

- if $s\in$ *Atoms*, then on the cell of the tape s will be written;
- if s is a list, then on the tape the beginning and the end of the list will be denoted by the symbols [and] correspondingly.

For example, the value (2 (4 ())76) will be represented on the tape as follows:

–	[	2	[	4	[	]	]	76	]	–

Let f: $B\rightarrow S$ be a computable function defined on *S-expression*. Let T be a Turing machine, which corresponds to function f. The finite set of the quadruples of T is the following: $\{q_{j_1}d_{i_1}d_{k_1}q_{r_1},\ q_{j_2}d_{i_2}d_{k_2}q_{r_2},\ldots,\ q_{j_m}d_{i_m}d_{k_m}q_{r_m}$, where $q_{j_l},\ q_{r_l}\in Q$, $d_{i_l}\in U$, $d_{k_l}\in U\cup\{L, R\}$, $l=1,\ldots,m$, $m\geq 1$. We will define a program P of Backus FP system, which models the work of the Turing machine T. For the program P the tape of the Turing machine will be presented as a list consisting of two linear lists. In the first list will be stored the symbols, which are on the left side of the head of the Turing machine, in the second list – which are on the right side. The first element of the second list will show the position of the head. For example, let the head of the Turing machine be placed on the cell of symbol d_6:

–	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	–
						△					

then the tape will be represented as follows: $((d_1\ d_2\ d_3\ d_4\ d_5\ )(d_6\ d_7\ d_8\ d_9\ d_{10}))$.

If the head of the machine shown above is moved to right, then the tape will be represented as follows: $((d_1\ d_2\ d_3\ d_4\ d_5\ d_6)(d_7\ d_8\ d_9\ d_{10}))$.

Let us define a program P, which corresponds to the Turing machine T:

$$F_1=comp(F_{decode},comp(F_T,constr(const(0),comp(apndl,constr(const(nil),$$
$$comp(apndl,constr(F_{encode,}const(nil)))))))$$
$$F_{null}=comp(eq,constr(id,const(nil)))$$
$$F_{not}=cond(id,const(F),const(T))$$
$$F_{and}=cond(hd,cond(comp(hd,tl),const(T),const(F),const(\mathrm{F}))$$
$$F_{atom}=cond(comp(eq,constr(id,\ const(a_1))),const(T),$$
$$\ldots$$
$$cond(comp(eq,constr(id,const(a_n))),const(T),const(F)))\ldots)$$
$$F_{apndr}=cond(comp(F_{null},hd),comp(apndl,constr(comp(hd,tl),const(nil))),$$
$$comp(apndl,constr(comp(hd,hd),comp(F_{apndr},constr(comp(tl,hd),comp(hd,tl))))))$$
$$F_{isLinear}=cond(F_{null},const(T),cond(comp(F_{and},constr(comp(F_{atom},hd),$$
$$comp(F_{not},comp(F_{null},hd)))),comp(F_{isLinear},tl),const(F)))$$
$$F_{conc}=cond(comp(F_{null},hd),comp(hd,tl),comp(apndl,constr(comp(hd,hd),$$
$$comp(F_{conc},\ constr(comp(tl,hd),\ comp(hd,tl))))))$$
$$F_{appndb}=comp(apndl,constr(const([),comp(F_{apndr},constr(id,const(])))))$$
$$F_{changeFirst}=cond(comp(F_{and},constr(comp(F_{atom},hd),comp(F_{not},comp(F_{null},hd)))),$$
$$comp(apndl,constr(hd,comp(F_{changeFirst},tl))),cond(comp(F_{isLinear},hd),$$
$$comp(F_{conc},constr(comp(F_{appndb},hd),\ tl)),comp(apndl,constr(comp(F_{changeFirst},hd),tl)))$$

$F_{listToTape}$=*cond*(*comp*($F_{isLinear}$,*id*),*comp*(F_{appndb},*id*),*comp*($F_{listToTape}$,(*comp*($F_{changeFirst}$,*id*)))
F_{encode}=*cond*(F_{atom},*id*,$F_{listToTape}$)
$F_{getLast}$=*cond*(*comp*(F_{null},*tl*),*hd*,*comp*($F_{getLast}$,*tl*))
$F_{exceptLast}$=*cond*(*comp*(F_{null},*tl*),*const*(*nil*),*comp*(*apndl*,*constr*(*hd*,*comp*($F_{exceptLast}$,*tl*))))
$F_{newTape}$=*cond*(*comp*(*eq*,*constr*(*hd*,*const*(*L*))),*cond*(*comp*(F_{null},*comp*(*hd*,*comp*(*hd*,*tl*))),*comp*(*apndl*,
constr(*const*(*nil*),*comp*(*apndl*,*constr*(*comp*(*apndl*,*constr*(*const*(–),*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*))))),
const(–))))),*cond*(*comp*(F_{and},*constr*(*comp*(F_{null},*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),*comp*(*eq*,*constr*(*comp*
($F_{getLast}$,*comp*(*hd*,*comp*(*hd*,*tl*))),*const*(–))))),*comp*(*apndl*,*constr*(*comp*($F_{exceptLast}$,*comp*(*hd*,*comp*(*hd*,*tl*))),
const((*nil*)))),*cond*(*comp*(F_{and},*constr*(*comp*(*eq*,*constr*(*comp*(*hd*,*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),*const*(–))),
comp(F_{null},*comp*(*tl*,*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*))))))),*comp*(*apndl*,*constr*(*comp*($F_{exceptLast}$,*comp*(*hd*,
comp(*hd*,*tl*))), *comp*(*apndl*,*constr*(*comp*(*apndl*,*constr*(*comp*($F_{gettLast}$,*comp*(*hd*,*comp*(*hd*,*tl*))),*const*(*nil*))),
const(*nil*))))),*comp*(*apndl*,*constr*(*comp*($F_{exceptLast}$,*comp*(*hd*,*comp*(*hd*,*tl*))),*comp*(*apndl,constr*(*comp*(*apndl*,
contr(*comp*($F_{gettLast}$,*comp*(*hd*,*comp*(*hd*,*tl*))),*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*))))),*const*(*nil*)))))))))),*cond*(*comp*
(*eq*,*constr*(*hd*,*const*(*R*))),*cond*(*comp*(F_{and},*constr*(*comp*(F_{null},*comp*(*hd*,*comp*(*hd*,*tl*))),*comp*(*eq*,*constr*(*comp*
(*hd*,*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),*const*(–))))),*comp*(*apndl*,*constr*(*const*(*nil*),*comp*(*apndl*,*constr*(*comp*(*tl*,
comp(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),*const*(*nil*))))),*cond*(*comp*(F_{null},*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),
comp(*apndl*,*constr*(*comp*(F_{apndr},*constr*(*comp*(*hd*,*comp*(*hd*,*tl*)),*const*(–))),*const*((*nil*)))),*comp*(*apndl*,
constr(*comp*(F_{apndr},*constr*(*comp*(*hd*,*comp*(*hd*,*tl*)),*comp*(*hd*,*comp*(*hd*,comp(*tl*,*comp*(*hd*,tl)))))),
comp(*apndl*,*constr*(*comp*(*tl*,*comp*(*hd*,*comp*(*hd*,*comp*(*tl*,*comp*(*hd*,*tl*)))),*const*(*nil*))))))),*comp*(*apndl*,
constr(*comp*(*hd*,*comp*(*hd*,*tl*)),*comp*(*apndl*,*constr*(*comp*(*apndl*,*constr*(*hd*,*comp*(*tl*,*comp*(*hd*,*comp*(*tl*,
comp(*hd*,*tl*)))))),*const*(*nil*))))))
F_T=*cond*(*comp*(F_{and},*constr*(*comp*(*eq*,*constr*(*hd*,*constr*(q_{j_1}))),*comp*(*eq*,*constr*(*comp*(*hd*,*comp*(*hd*,*comp*(*tl*,
comp(*hd*,*tl*)))),*const*(d_{i_1}))))),*comp*(F_T,*constr*(*const*(q_{r_1}),*comp*($F_{newTape}$,*constr*(*const*(d_{k_1}),*hd*)))),

…

cond(*comp*(F_{and},*constr*(*comp*(*eq*,*constr*(*hd*,*constr*(q_{j_m}))),*comp*(*eq*,*constr*(*comp*(*hd*,*comp*(*hd*,*comp*(*tl*,
comp(*hd*,*tl*)))),*const*($d_{i_{\mathrm{m}}}$))))),*comp*(F_T,*constr*(*const*(q_{r_m}),*comp*($F_{newTape}$,*constr*(*const*(d_{k_m}),*hd*)))),
comp(*hd*,*comp*(*tl*, *comp*(*hd*,*tl*)))…)
$F_{ttlMakeList}$=*cond*(*comp*(F_{null},*hd*),*const*(*nil*),*cond*(*comp*(*eq*,*constr*(*comp*(*hd*,*hd*),*const*([))),
comp(F_{conc},*constr*(*comp*(*apndl*,*constr*(*comp*(*hd*,*tl*),*const*(*nil*))),*comp*(*tl*,*hd*))),*comp*($F_{ttlMakeList}$,*constr*
(*comp*(*tl*,*hd*),*comp*(*apndl*, *constr*(*comp*(*hd*,*hd*), *comp*(*hd*,*tl*)))))))
$F_{tapeToList}$=*cond*(*comp*(F_{null},*hd)*,*comp*(*hd*,*comp*(*hd*,*tl*)),*cond*(*comp*(*eq*,*constr*(*comp*(*hd*,*hd*),*const*(]))),
comp($F_{tapeToList}$,*constr*(*comp*(*tl*,*hd*),*comp*($F_{ttlMakeList}$,*constr*(*comp*(*hd*,*tl*),*const*(*nil*))))),*comp*($F_{tapeToList}$,
constr(*comp*(*tl*,*hd*),*comp*(*apndl*,*constr*(*comp*(*hd*,*hd*),*comp*(*hd*,*tl*)))))))
F_{decode}=*cond*(*comp*(*eq*,*constr*(*hd*,*const*([))),*comp*($F_{tapeToList}$,*constr*(*id*,*const(nil*))),*id*)

The program P consists of 19 equations. For this program we constract a sequence of programs P_0, P_1,…,P_{18} such that P_0 is the program P, P_{18} consists of one equation, the program P_{i+1} is obtained from P_i by the following steps ($i = 0,\dots,17$):

1) remove the second equation of P_i;

2) in the rest of equations of P_i replace all free occurrences of the variable of the left side of the second equation with the least solution of that equation.

Applying Theorem 2.1 of [5], we get $f_P = f_{P_j}$ for any $j = 0,\dots,18$. In each program P_i, $i=0,\dots,17$, the right side of the second equation has at most one free variable, which is the same as the variable of the left side. So, that equation can be solved independently from the other equations of P_i.

Theorem 3.1 is proved by the construction of the sequence P_0, P_1,…,P_{18}. The fixpoint semantics of the program P is the following: $f_P(m) = \begin{cases} f(m), & \text{if } m \in B, \\ \perp, & \text{otherwise,} \end{cases}$ where $m \in M$.

Theorem 3.2. The set of built-in constants $C_1 \cup C_2$ is minimal for Backus FP system, which uses more than three atoms.

Theorem 3.3. For Backus FP system, which uses only three atoms, we have:

a) the set of built-in constants $C_1 \cup C_2 \backslash \{const\}$ is complete and minimal;

b) for any constant $\varphi \in C_1 \cup C_2 \backslash \{const\}$ the set of built-in constants $C_1 \cup C_2 \backslash \{\varphi\}$ is not complete.

The proofs of Theorems 3.2 and 3.3 will be deduced using the following lemmas.

Lemma 3.1. The function *id* is not representable in Backus FP system, which uses the set of constants $M \cup C_1 \cup C_2 \backslash \{id\}$.

We will show that any function *f*, which is representable in Backus FP system, which uses the set of constants $M \cup C_1 \cup C_2 \backslash \{id\}$, is a constant function on the set *Atoms*, i.e. for any $m \in Atoms$, $f(m) = c$, $c \in M$. So, the function *id* is not representable in that system, bacause it is not a constant function on the set *Atoms*.

If $f \in \{hd, tl, apndl, eq\}$, then $f(m) = \perp$ for any $m \in Atoms$. So, the function *f* is constant on the set *Atoms*.

Let the functions *g*, *h*, *p* be constant on the set *Atoms*, $g(m)=c$, $h(m)=c'$, $p(m)=c''$, where $c,c',c'' \in M$ for all $m \in Atoms$. Let us show that the functions $comp(g,h)$, $constr(g,h)$, $const(m_0)$ and $cond(p,g,h)$ are also constant functions on the set *Atoms*, where $m_0 \in M$.

For any $m \in Atoms$, $comp(g,h)(m) = g(h(m)) = g(c') = m_0$, $m_0 \in M$, because $h(m)=c'$, $c' \in M$, for any $m \in Atoms$.

For any $m \in Atoms$, $constr(g,h)(m) = (g(m)\ h(m)) = (c\ c') \in M$, since $g(m) = c$, $h(m) = c'$, $c,c' \in M$, for any $m \in Atoms$.

For any $m \in Atoms$, $const(m_0)(m) = m_0 \in M$.

For any $m \in Atoms$, we have that $p(m) = c''$, $c'' \in M$. If $p(m)=T$, then $cond(p, g, h)(m) = g(m) = c$, $c \in M$, since the function *g* is constant on the set *Atoms*. If $p(m) = F$, then $cond(p, g, h)(m) = h(m) = c'$, $c' \in M$, because the function *h* is a constant function on the set *Atoms*. If $p(m) \neq T$, $p(m) \neq F$, then $cond(p, g, h)(m) = \perp$.

Let P_1 be a program in Backus FP system, which uses the set of constants $M \cup (C_1 \cup C_2 \backslash \{id\})$. Let us show that the function $(\Psi_{P_1}^k(\bar{\Omega}))_i$ is a constant function on the set *Atoms* for any $k \geq 0$ and $1 \leq i \leq n$, $n \geq 1$. In the case $k = 0$ for any $1 \leq i \leq n$, $n \geq 1$, and $m \in M$, $(\Psi_{P_1}^0(\bar{\Omega}))_i(m) = (\bar{\Omega})_i(m) = \perp$. So, the function $(\Psi_{P_1}^0(\bar{\Omega}))_i$ is constant on the set *Atoms*.

Let us assume that the function $(\Psi_{P_1}^k(\bar{\Omega}))_i$ is constant on the set *Atoms* for any $k \geq 0$. We will show that for any $1 \leq i \leq n$, $n \geq 1$ the function $(\Psi_{P_1}^{k+1}(\bar{\Omega}))_i$ is also a constant function on the set *Atoms*. The function $(\Psi_{P_1}^{k+1}(\bar{\Omega}))_i$ is obtained from the functions $(\Psi_{P_1}^k(\bar{\Omega}))_1, \ldots, (\Psi_{P_1}^k(\bar{\Omega}))_n$ and the constants C_1, so, is a constant function on the set *Atoms*.

Now let us show that the function f_{P_1} is constant on the set *Atoms*. We will proceed by contradiction. Let us assume that f_{P_1} is not constant on the set *Atoms*, so, there exist m_1, $m_2 \in Atoms$, such that $f_{P_1}(m_1) \neq f_{P_1}(m_2)$. This means that there exist $j_1, j_2 \geq 0$ such that $f_{P_1}(m_1) = (\Psi_{P_1}^{j_1}(\bar{\Omega}))_1(m_1)$, $f_{P_1}(m_2) = (\Psi_{P_1}^{j_2}(\bar{\Omega}))_1(m_2)$. Let $j = \max(j_1, j_2)$. Then $(\Psi_{P_1}^j(\bar{\Omega}))_1(m_1) \neq$ $\neq (\Psi_{P_1}^j(\bar{\Omega}))_1(m_2)$, so, the function $(\Psi_{P_1}^j(\bar{\Omega}))_1$ is not constant on the set *Atoms*, which is a contradiction.

The Lemma 3.1 is proved.

The proofs of the following Lemmas are similar to the proof of the Lemma 3.1. The main idea is to show that any function, which is representable in Backus FP system, which uses a given set of built-in constants, has some properties, but there exists a computable function, which does not have any of those properties, so, it is not reproducable in Backus FP system, that uses a given set of built-in constants.

Lemma 3.2. The function *hd* is not representable in Backus FP system, which uses the set of constants $M\cup(C_1\cup C_2\setminus\{hd\})$.

Let us define the change of underlined $\underline{m}$ by m' in a term t (denoted by $t\{\underline{m}\Rightarrow m'\}$), where $t\in\Lambda$, $m, m'\in M$:

1. if $t\equiv c$, $c\in\alpha$, $\alpha\in Types$, then
   1.1. if $t\equiv m$ and m is underlined, then m';
   1.2. if $t\equiv(s_1\ldots s_n)$, $s_i\in M$, $n\geq 0$, then $t\{\underline{m}\Rightarrow m'\}\equiv(s_1\{\underline{m}\Rightarrow m'\}\ldots s_n\{\underline{m}\Rightarrow m'\})$;
2. if $t\equiv x$, $x\in V$, then t;
3. if $t\equiv\tau(t_1,\ldots,t_k)$ $x\in\Lambda_\beta$, $\tau\in\Lambda_{[\alpha_1\times\ldots\times\alpha_k\to\beta]}$, $t_i\in\Lambda_{\alpha_i}$, $\alpha_i,\beta\in Types$, $i=1,\ldots,k$, $k\geq 1$, then

$\tau(t_1,\ldots,t_k)\{\underline{m}\Rightarrow m'\}\equiv\tau\{\underline{m}\Rightarrow m'\}(t_1\{\underline{m}\Rightarrow m'\},\ldots,t_k\{\underline{m}\Rightarrow m'\})$;

We will say that term t' is obtained from term $t(t, t'\in\Lambda_\alpha, \alpha\in Types)$ by changing $\underline{m}_1$ by $m'_1,\ldots, \underline{m}_n$ by m'_n (denoted by $t\{\underline{m}_1\Rightarrow m'_1,\ldots,\underline{m}_n\Rightarrow m'_n\}\equiv t'$), where $\underline{m}_i, m'_i\in M$, $i\neq j\Rightarrow \underline{m}_i\neq\underline{m}_j$, $i, j=1,\ldots,n$, $n\geq 1$, if there exist terms $t_0,\ldots,t_n\in\Lambda_\alpha$ such that $t\equiv t_0$, $t'\equiv t_n$ and $t_n\{\underline{m}_i\Rightarrow m'_i\}\equiv t_{i+1}$, $i=0,\ldots,n-1$, $n\geq 1$.

Let $m\in S$ be a nonempty list. Let us underline one or more subobjects of m (not coinciding with m) such that there is no underlined subobject, which is subobject of underlined subobject. By $m_0\in S$ we will denote the object, which is obtained from m by replacing all underlined subobjects with $\underline{m'_0}$, where $m'_0\in S$. $m'_1,\ldots,m'_n,\ldots\in S$ $(n\geq 1)$ is a countable set of non-coinciding objects. By M_{m_0} we will denote the following set of objects: $M_{m_0}=\{m_i|\ m_0\{\underline{m'_0}\Rightarrow\underline{m'_i}\}\equiv m_i, i\geq 1\}$.

Let us fix a set $M_{m_0}(m_0\in S)$. We will say that the function f is almost everywhere constant on the set M_{m_0}, if there exist finite number of elements $m_{i_1},\ldots,m_{i_n}\in M_{m_0}$, $n\geq 0$, such that: $f(m)=c$, $c\in M$ for any $m\in M_{m_0}\setminus\{m_{i_1},\ldots,m_{i_n}\}$.

We will say that the function f almost everywhereon the set M_{m_0} returns a list containing the underlined object, if there exist finite number of elements $m_{i_1},\ldots,m_{i_n}\in M_{m_0}$, $n\geq 0$, such that: $f(m)$ is a list containing the underlined object, for any $m\in M_{m_0}\setminus\{m_{i_1},\ldots,m_{i_n}\}$.

Here any function f which is representable in Backus FP system, which uses the set of constants $M\cup(C_1\cup C_2\setminus\{hd\})$, has one of the following properties: 1) for any $M_{m_0}(m_0\in S)$ the function f is almost everywhere constant on the set M_{m_0}; 2) for any $M_{m_0}(m_0\in S)$ the function f almost everywhere on the set M_{m_0} returns a list containing the underlined object. Let us show that the function *hd* does not have any of these properties. Let $M_{m_0}=\{(\underline{m_i})|\ m_i\in M,\ i\neq j\Rightarrow m_i\neq m_j,\ i,j\geq 0\}$. Then for any $i\geq 0$, $hd((m_i))=\underline{m_i}$.

The Lemma 3.2 is proved.

Lemma 3.3. The function *tl* is not representable in Backus FP system, which uses the set of constants $M\cup(C_1\cup C_2\setminus\{tl\})$.

One can prove that any function f, which is representable in Backus FP system that uses the set of constants $M\cup(C_1\cup C_2\setminus\{tl\})$, has one of the following properties: 1) for any $M_{m_0}(m_0\in S)$ the function f is almost everywhere constant on the set M_{m_0}; 2) for any $M_{m_0}(m_0\in S)$ the function f almost everywhere on the set M_{m_0} returns a list containing the underlined object and there is no sublist whith first element underlined. Let us show that the function *tl* does not have any of these properties.

Let $M_{m_0}=\{(T\ \underline{m_i})|\ m_i\in M,\ i\neq j\Rightarrow m_i\neq m_j,\ i,j\geq 0\}$.

For any $i \geq 0$ $tl((T\underline{m_i})) = (\underline{m_i})$.

The Lemma 3.3 is proved.

Lemma 3.4. The function *apndl* is not representable in Backus FP system, which uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{apndl\})$.

Let $m \in M$. By $H(m)$ we will denote the length of the longest linear sublist of m (if $m \in Atoms$ or $m = \perp$, then $H(m) = 0$).

Here it be proven that for any function f, which is representable in Backus FP system that uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{apndl\})$, there exists a number $d_f \geq 3$, such that if $H(m) \leq d_f$, $m \in M$, then $H(f(m)) \leq d_f$, and if $H(m) > d_f$, then $H(f(m)) \leq H(m)$. It's obvious the function *apndl* does not have this property, since for any $d \geq 0$, if $m = (T(\underbrace{T \ldots T}_{d}))$, then we have $H(m) = d$, but $H(apndl(m)) = d + 1$.

The Lemma 3.4 is proved.

Lemma 3.5. The function *eq* is not representable in Backus FP system, which uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{eq\})$.

Here one can show that any function f, which is representable in Backus FP system that uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{eq\})$, has one of the following properties: for any $m, m' \in M$, satisfying $m\{\underline{T} \Rightarrow nil\} = m'$; 1) $f(m)\{\underline{T} \Rightarrow nil\} = f(m')$; 2) $f(m) = \perp$ or $f(m') = \perp$; 3) $f(m) = f(m')$. Let us show that the function *eq* does not have any of these properties. Let $m = (\underline{T}\ T)$, $m' = (nil\ T)$. In this case $eq(m) = T$, $eq(m') = F$.

The Lemma 3.5 is proved.

Lemma 3.6. The set $C_1 \cup C_2 \setminus \{comp\}$ is not complete.

In this case one can show that for any function f, which is representable in Backus FP system that uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{comp\})$, there exists a number $d_f \geq 0$, such that if $m \in M$ is a linear list with a length $n \geq d_f$, then $f(m)$ cannot be a linear list with a length greater than n.

Let us consider the following function: $f_{comp}(m) = \begin{cases} (T\,m_1 \ldots m_n), \text{ if } m = (m_1 \ldots m_n),\ n \geq 0, \\ \perp, \text{ otherwise.} \end{cases}$

It's obvious that the function f_{comp} does not have this property, because for any $d \geq 0$ if $m \in M$ is a linear list with a length $n \geq d$, then $f_{comp}(m)$ is a linear list with a length $n + 1$. So, the set $C_1 \cup C_2 \setminus \{comp\}$ is not complete.

The Lemma 3.6 is proved.

Lemma 3.7. The set $C_1 \cup C_2 \setminus \{constr\}$ is not complete.

Let $m \in M$. By $D(m)$ we will denote the depth of m:

1. if $m \in Atoms$ or $m = \perp$, then $D(m) = 0$;
2. if $m = (m_1 \ldots m_n)$, $m_i \in S$, $i = 1, \ldots, n$, $n \geq 0$, then $D(m) = \max(D(m_1), \ldots, D(m_n)) + 1$.

Here it can be proved that for any function f, which is representable in Backus FP system that uses the set of constants $M \cup (C_1 \cup C_2 \setminus \{constr\})$, there exists a number $d_f \geq 0$, such that if $D(m) \leq d_f$, $m \in M$, then $D(f(m)) \leq d_f$, and if $D(m) > d_f$, then $D(f(m)) \leq D(m)$.

Let us consider the following function: $f_{constr}(m) = \begin{cases} (m), \text{ if } m \neq \perp, \\ \perp, \text{ otherwise.} \end{cases}$

It's obvious that the function f_{constr} does not have the mentioned property, because for any $d \geq 0$, if $m = \underbrace{((\ldots(}_{d}T)\ldots)$, we will have $D(m) = d$, but $D(f_{constr}(m)) = d + 1$. So, the set $C_1 \cup C_2 \setminus \{constr\}$ is not complete.

The Lemma 3.7 is proved.

Lemma 3.8. The set $C_1 \cup C_2 \backslash \{cond\}$ is not complete.

It can be proved that any function f, which is representable in Backus FP system that uses the set of constants $M \cup (C_1 \cup C_2 \backslash \{cond\})$, has one of the following properties: for any $m, m' \in M$ such that $m\{\underline{T} \Rightarrow nil, \underline{T} \Rightarrow F, \underline{F} \Rightarrow T\} = m'$:
1) $f(m)\{\underline{T} \Rightarrow nil, \underline{T} \Rightarrow F, \underline{F} \Rightarrow T\} = f(m')$; 2) $f(m) = \perp$ or $f(m') = \perp$; 3) $f(m) = f(m')$.

Let us consider the following function: $f_{cond}(m) = \begin{cases} nil, \text{ if } m = T, \\ T, \text{ if } m = nil, \\ \perp, \text{ otherwise.} \end{cases}$

It's obvious that the function f_{cond} does not have any of these properties, because $f_{cond}(\underline{T}) = nil$, $f_{cond}(nil) = T$. So, the set $C_1 \cup C_2 \backslash \{cond\})$ is not complete.

The Lemma 3.8 is proved.

It is obvious that the Lemmas 3.1–3.8 remain true for Backus FP system which uses three or more atoms.

Lemma 3.9. The set $C_1 \cup C_2 \backslash \{const\}$ is not complete for Backus FP system, which uses more than three atoms, but it is complete for Backus FP system, which uses exactlly three atoms.

Let us show that the functions $const(T)$, $const(F)$ and $const(nil)$ are representable in Backus FP system, which uses only three atoms and the set of constants $M \cup (C_1 \cup C_2 \backslash \{const\})$. It is evident,

$const(T) = comp(eq, constr(id, id))$,

$const(F) = comp(eq, constr(id, constr(id, id)))$,

$const(nil) = comp(tl, comp(tl, constr(id, id)))$.

All other constant functions can be obtained by these functions and the constants *apndl*, *comp*, *constr*.

Now let us consider Backus FP system ,which uses more than three atoms and the set of constants $M \cup (C_1 \cup C_2 \backslash \{const\})$, i.e. $\{a, T, F, nil\} \subset Atoms$. By M_a we will denote the set of all objects is obtained from M by removing the atom a and all lists containing the atom a.

Here one can show that for any function f, which is representable in Backus FP system that uses more than three atoms and the set of constants $M \cup (C_1 \cup C_2 \backslash \{const\})$, if $m \in M_\alpha$, then $f(m) \in M_\alpha$.

Let us consider the following function: $f_{const}(m) = \begin{cases} a, \text{ if } m \neq \perp, \\ \perp, \text{ otherwise.} \end{cases}$

It's obvious that the function f_{const} does not have this property, because for any $m \in M_\alpha$, $f_{const}(m) \notin M_\alpha$. So, the set $C_1 \cup C_2 \backslash \{const\}$ is not complete.

The Lemma 3.9 is proved.

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Բեկուսի FP համակարգի ներդրված հաստատունների լրիվ և մինիմալ բազմության մասին

Աշխատանքում ապացուցված է Բեկուսի FP համակարգի ներդրված հաստատունների հետևյալ բազմության լրիվությունը և մինիմալությունը. *Identity, Head, Tail, Append left, Equals, Composition, Construction, Condition, Constant*:

Об одном полном и минимальном множестве встроенных констант системы Бэкуса FP

В данной работе доказана полнота и минимальность множества следующих встроенных констант: *Identity, Head, Tail, Append left, Equals, Composition, Construction, Condition, Constant.*