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Mathematics

ON GENERALIZATION OF THE THEOREM OF PICARD

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The present article is devoted to the generalization of the classic theorem of Picard on entire functions in case of $\mathcal{A}$-valued analitycal functions.

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Let $\mathcal{A}$ be a complex Banach algebra with unit $\mathbf{1}$ (we assume that $\|\mathbf{1}\| = 1$ and $\|xy\| \leq \|x\| \cdot \|y\|$ for every $x, y \in \mathcal{A}$).

Suppose simultaneously, a locally convex topology τ is given on the algebra $\mathcal{A}$, for which the identical map $(\mathcal{A}, \|\cdot\|) \to (\mathcal{A}, \tau)$ is continuous and multiplication is τ-continuous with respect to each component (see [1–3]). Let $\{P_\alpha\}_{\alpha\in\Gamma}$ be the system of algebraic seminorms, defining τ and assume that topological algebra $(\mathcal{A}, \tau)$ is complete, locally convex. Let J be a closed two-sided ideal in algebra $(\mathcal{A}, \tau)$. Then the factor algebra $(\mathcal{A}, \tau)/_J$ is complete and locally convex with respect to factor topology τ_Φ generated by the corresponding family of algebraic factor seminorms $\{q_\gamma\}_{\gamma\in\Gamma}$. Note that the factor topology τ_Φ is the strongest topology in the algebra $(\mathcal{A}, \tau)/_J$, where the canonical homomorphism

$$\pi_{\mathcal{J}} : (\mathcal{A}, \tau) \to (\mathcal{A}, \tau)/_J$$

is continuous.

Recall (see [4, 5]) that in the complex algebra $\mathcal{A}$ with unit, a linear operator $\mathcal{D} : \mathcal{A} \to \mathcal{A}$, satisfying

$$\mathcal{D}(xy) = x(\mathcal{D}y) + (\mathcal{D}x)y \quad (x, y \in \mathcal{A}),$$

is called $\mathcal{A}$-differentiation.

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Denote by $\mathrm{Der}\big((\mathcal{A},\tau)\big)$ the set of all continuous $(\mathcal{A},\tau)$-differentiations. Assume that $\mathcal{D}\in \mathrm{Der}\big((\mathcal{A},\tau)\big)$ and J is a closed, two-sided ideal. Then the continuous $(\mathcal{A},\tau)\big/_J$ -differentation, given by the formula

$$\mathcal{D}_J,(\hat{a}) = \pi_J(\mathcal{D}a),$$

acts on the factor algebra $(\mathcal{A},\tau)\big/_J$.

Note that the set $V(a)=\left\{\varphi(a) : \|\varphi\| = \varphi(\mathbf{1}) = 1, \varphi\in\mathcal{A}^*\right\}$ is called the algebraic numerical range of the element $a\in\mathcal{A}$.

The following abstract version of the classical Picard theorem on entire functions is true (see [5,6]).

Theorem. Let $(\mathcal{A},\tau)$ be a complex, complete, locally convex algebra with unit, and J be a closed, two-sided ideal in the algebra $(\mathcal{A},\tau)$ and $\mathcal{D}\in\mathrm{Der}\big((\mathcal{A},\tau)\big)$. If the element $a\in(\mathcal{A},\tau)$ is such that $\mathcal{D}a\notin J$, then

$$\bigcup_{\lambda\in\mathbb{C}} V\Big(\big(\exp(\lambda\mathcal{D})\big)_J(\hat{a})\Big) = \mathbb{C}.$$

Proof. For every $j\in J$ we have $\mathcal{D}a+j\notin J$, since $\mathcal{D}a\notin J$. Hence, we have that $\exp\lambda(\mathcal{D}a+j)\notin J$.

Suppose that $U_j = \bigcup\limits_{\lambda\in\mathbb{C}} U_\lambda^{(j)}$, where $U_\lambda^{(j)} = V(\exp\lambda(\mathcal{D}a+j))$. Let us prove, that $U_j=\mathbb{C}$. Let $\mathbb{C}\setminus U_j$ contains at least two points. For $\varphi\in St((\mathcal{A},\tau))$ the image of the entire function

$$f_\varphi^{\langle j\rangle}(\lambda) = \varphi(\exp\lambda(\mathcal{D}a+j))$$

is in the set U_j. Therefore, according to the classical Picard theorem for entire functions, we have that (see [7]) $f_\varphi^{\langle j\rangle}(\lambda)\equiv$const. Then $f_\varphi^{\langle j\rangle}(\lambda)\equiv 0$ implying $\mathcal{D}a+j=0$. Consequently, $\mathcal{D}a=-j\in J$, which contradicts the Theorem conditions.

Thus, the set $\mathbb{C}\setminus U_j$ can contain at most one point. Let us show that $U_j=\mathbb{C}$.

Let $\xi_0\in sp(\mathcal{D}a+j)$ and $\xi\in\mathbb{C}$ be any complex number. Then there exists a point $\xi_1\in U_\lambda^{\langle j\rangle}$ such that ξ is on the interval $[\xi_0,\xi_1]$. Since $\xi_0\in U_\lambda^{\langle j\rangle} =$ $= V(\exp\lambda(\mathcal{D}a+j))$ and $U_\lambda^{\langle j\rangle}$ is convex, then $\xi\in U_\lambda^{\langle j\rangle}$, and therefore, $\xi\in U_j$. Thus, we have $U_j=\mathbb{C}$ for every $j\in J$. Consequently, the equality

$$V\big((\exp(\lambda\mathcal{D}))_{\mathcal{J}}(\hat{a})\big) = \bigcap_{j\in(J,\tau)} V(\exp\lambda(\mathcal{D}a+j)) = \bigcap_{j\in J} U_j$$

is true in the factor algebra $(\mathcal{A},\tau)\big/_J$ for every $j\in\mathcal{J}$, $U_j=\mathbb{C}$. As a result we have that

$$V\left((\exp(\lambda\mathcal{D}))_J(\hat{a})\right)=\mathbb{C}. \qquad \square$$

In case when the topology τ coincides with the norm topology, we will get:

Corollary 1. Let $\mathcal{A}$ be a complex Banach algebra with unit, and J is a closed, two-sided ideal in $\mathcal{A}$ and $\mathcal{D}\in\mathrm{Der}(\mathcal{A})$. If the element $a\in\mathcal{A}$ is such that $\mathcal{D}a\notin J$, then

$$\bigcup_{j=\mathbb{C}} V\big((\exp(\lambda\mathcal{D}))_{\mathcal{J}}(\hat{a})\big)=\mathbb{C}.$$

In case when $J = \{0\}$ Corollary 1 implies:

C o r o l l a r y 2. Let $\mathcal{A}$ be a complex Banach algebra with unit and $\mathcal{D} \in \mathrm{Der}(\mathcal{A})$. If the element $a \in \mathrm{Ker}(\mathcal{D})$, then

$$\bigcup_{j\in\mathbb{C}} V\left(\left(\exp(\lambda\mathcal{D})\right)(a)\right) = \mathbb{C}.$$

For inner differentiations we have:

C o r o l l a r y 3. Let $\mathcal{A}$ be a complex Banach algebra with unit, $\mathcal{J}$ be a closed two-sided ideal. If the elements $a,b,c \in \mathcal{A}$ are such that $ac \neq cb(\mod \mathcal{J})$, then

$$\bigcup_{j\in\mathbb{C}} V\left((\exp\lambda\hat{a})\,\hat{c}\exp\left(-\lambda\hat{b}\right)\right) = \mathbb{C}.$$

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R E F E R E N C E S

1. **Williams J.P.** On Commutativity and Numerical Range in Banach Algebras. // Journal of Functional Analysis, 1972, v. 10, p. 326–323.
2. **Karakhanyan M.I.** An Asymptotic Version of General Commutator Theorems. // Functional Anal. i Prilozhen, 2005, v. 39, № 4, p. 80–83 (in Russian).
3. **Karakhanyan M.I., Kamalyan H.A.** A Remark on Commutativity of the Image of $\mathcal{A}$-Valued Analytic Function. // Izvestia NAN Armenii. Matematika, 2013, v. 48, № 2, p. 39–42.
4. **Helemski A.Ya.** The Homology of Banach and Topological Algebras. M.: Izd. MGU, 1986 (in Russian).
5. **Bonsall J.F., Dunkan J.** Complete Normed Algebras. Berlin–Heidelberg–New-York: Springer-Verlag, 1973.
6. **Stoilov S.** Theory of Functions of a Complex Variable. M.: Inostr. Literatura, 1962, v. 2 (in Russian).
7. **Karakhanyan M.I.** On the Abstract Theorem of Picard. // Lobachevskii J. Math., 2005, v. 18, p. 127–130.