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Mathematics

ON RECOVERY OF A FRANKLIN SERIES FROM ITS SUM

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In an earlier work of author a theorem on recovery of Franklin series from its sum under some conditions was obtained. In present article it is shown that to some extent these conditions are necessary.

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Introduction. Recall the definition of Franklin system. Let $n = 2^k + i$, where $k \geq 0$ and $1 \leq i \leq 2^k$. Denote

$$s_{n,j} = \begin{cases} j/2^{k+1} & \text{for} \quad 0 \leq j \leq 2i, \\ (j-2i)/2^k & \text{for} \quad 2i+1 \leq j \leq n. \end{cases}$$

Let S_n be the space of continuous and piecewise linear functions on $[0;1]$ with grid points $\{s_{n,j}\}_{j=0}^{n}$, i.e. $f \in S_n$, if and only if $f \in C[0;1]$ and f is linear on each $[s_{n,j-1};s_{n,j}]$, $j = 1,2,\dots,n$. It is clear that $\dim S_n = n+1$ and the set $\{s_{n,j}\}_{j=0}^{n}$ is obtained from $\{s_{n-1,j}\}_{j=0}^{n-1}$ by adding the point $s_{n,2i-1}$. Therefore, there exists a unique up to sign function $f_n \in S_n$, which is orthogonal to S_{n-1} and $\|f_n\|_2 = 1$. Defining $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x-1)$, $x \in [0;1]$, we will obtain an orthonormal system $\{f_n(x)\}_{n=0}^{\infty}$, which was first introduced in [1].

Consider a series $\sum\limits_{n=0}^{\infty} a_n f_n(x)$. Denote by

$$\sigma_\nu(x) := \sum_{n=0}^{2^\nu} a_n f_n(x) \text{ and } \sigma^*(x) := \sup_\nu |\sigma_\nu(x)|.$$

$|A|$ denotes the Lebesgue measure of a set A.

Let functions $h_m(x) : [0,1] \to R$ satisfy the following conditions:

$$0 \leq h_1(x) \leq h_2(x) \leq \dots \leq h_m(x) \leq \dots, \lim_{m\to\infty} h_m(x) = \infty \tag{1}$$

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and there exists dyadic points $0 = t_{m,0} < t_{m,1} < \ldots < t_{m,s_m} = 1$ so that the intervals $I_k^m = [t_{m,k-1}, t_{m,k})$, $k = 1, \ldots, s_m$, are dyadic as well, i.e. I_k^m is of the form $\left[\frac{i}{2^j}, \frac{i+1}{2^j}\right)$, and the function $h_m(x)$ is constant on those intervals: $h_m(x) = \lambda_k^m$ for $x \in I_k^m$, $k = 1, \ldots, s_m$.

Moreover

$$\inf_{m,k} \int_{I_k^m} h_m(x)dx = \inf_{m,k} |I_k^m| \lambda_k^m > 0, \tag{2}$$

$$\sup_{m,k} \left(\frac{\lambda_k^m}{\lambda_{k-1}^m} + \frac{\lambda_{k-1}^m}{\lambda_k^m} \right) < +\infty \tag{3}$$

and

$$\sup_{m,k} \left(\frac{|I_k^m|}{|I_{k-1}^m|} + \frac{|I_{k-1}^m|}{|I_k^m|} \right) < +\infty. \tag{4}$$

In other words, for any function h_m the interval $[0,1]$ can be partitioned into dyadic intervals, so that the values of the function on neighbouring intervals are equivalent to each other and so are the lengths of neighbouring intervals. Moreover the integrals of h_m over these intervals are bounded away from 0.

The following theorem was proved in [2].

Theorem A. Let the sequence $h_m(x)$ satisfy conditions (1)–(3). If the partial sums $\sigma_\nu = \sum_{n=0}^{2^\nu} a_n f_n$ converge in measure to a function f, and the majorant σ^* of partial sums σ_ν satisfies

$$\lim_{m\to\infty} \int_{\{x\in[0,1];\ \sigma^*(x)>h_m(x)\}} h_m(x)dx = 0, \tag{5}$$

then for any $n \geq 0$

$$a_n = \lim_{m\to\infty} \int_0^1 [f(x)]_{h_m(x)} f_n(x)dx, \tag{6}$$

where $[f(x)_{g(x)}]$ is equal to $f(x)$, if $|f(x)| < g(x)$ and 0 otherwise.

A similar theorem for Haar series was proved in [3]. Theorem A is a generalization of an uniqueness theorem for Franklin series a.e. converging to 0, which was obtained in [4].

The main goal of this article is the following theorem, which shows the necessity of condition (3) to some extent in the Theorem A .

Theorem. Let functions $h_m(x)$ satisfy conditions (1), (2), (4) and

$$\sup_{m,k} \left(\frac{\lambda_k^m}{\lambda_{k-1}^m \cdot \log \lambda_k^m} + \frac{\lambda_{k-1}^m}{\lambda_k^m \cdot \log \lambda_{k-1}^m} \right) = +\infty. \tag{7}$$

Then there exists a series $\sum_{n=0}^{\infty} a_n f_n$, converging a.e. to some function f, with a majorant σ^* of partial sums σ_ν satisfying (5), but the coefficients a_n, $n \geq 0$, are not recovered by formulas (6), particularly

$$\limsup_{m\to\infty} \int_0^1 [f(x)]_{h_m(x)} f_0(x)dx = \limsup_{m\to\infty} \int_0^1 [f(x)]_{h_m(x)} dx = +\infty.$$

Some Properties of Franklin Functions. It follows from the definition of Franklin functions that each of them is linear on $[s_{n,j-1}, s_{n,j}]$. Therefore, the function $f_n(t)$ is uniquely determined by the values $a_j^{(n)} := f_n(s_{n,j})$. It is known that the coefficients $a_j^{(n)}$ satisfy the following inequalities (see, e.g., [5]):

$$\begin{cases} \frac{1}{4}\left|a_{j+1}^{(n)}\right| \le \left|a_j^{(n)}\right| \le \frac{2}{7}\left|a_{j+1}^{(n)}\right| & \text{for} \quad 1 \le j \le 2i-3, \\ \frac{1}{4}\left|a_{j-1}^{(n)}\right| \le \left|a_j^{(n)}\right| \le \frac{2}{7}\left|a_{j-1}^{(n)}\right| & \text{for} \quad 2i+1 \le j \le n-1. \end{cases} \tag{8}$$

$$\begin{cases} \left|a_{2i}^{(n)}\right| \le \frac{48}{97}\left|a_{2i-1}^{(n)}\right| = \frac{48}{97}a_{2i-1}^{(n)}, \\ \left|a_{2i-2}^{(n)}\right| \le \frac{66}{107}\left|a_{2i-1}^{(n)}\right|. \end{cases} \tag{9}$$

Moreover the coefficients $a_j^{(n)}$ are checkerboard, i.e.

$$(-1)^{j-1}a_j^{(n)} > 0, \tag{10}$$

$$\|f_n\|_p \sim n^{1/2-1/p} \quad \text{for} \quad 1 \le p \le \infty. \tag{11}$$

Note that it follows from (9) and (8) that

$$\|f_n\|_\infty = a_{2i-1}^{(n)}. \tag{12}$$

The aim of this section is to prove the following lemma.

Lemma. Let $n = 2^k + i$, where $k \ge 0$ and $5 \le i \le 2^k - 4$. There exists a constant c so that for any $n \ge 16$

$$\left|\int_0^{s_{n,2i-2}} f_n(t)dt\right| = \int_{s_{n,2i-2}}^1 f_n(t)dt \ge c\|f_n\|_1.$$

Proof. Since $\int_0^1 f_n(t)dt = \int_0^1 f_n(t)f_0(t)dt = 0$, hence, $\left|\int_0^{s_{n,2i-2}} f_n(t)dt\right| = \left|\int_{s_{n,2i-2}}^1 f_n(t)dt\right|$. Therefore, it is sufficient to prove the inequality.

For brevity we will omit the parameter n and will write s_j, a_j instead of $s_{n,j}$, $a_j^{(n)}$. Let us split the integral into three parts:

$$I := \int_{s_{2i-2}}^1 f_n(t)dt = \int_{s_{2i-2}}^{s_{2i+1}} f_n(t)dt + \int_{s_{2i+1}}^{s_{2i+3}} f_n(t)dt + \int_{s_{2i+3}}^1 f_n(t)dt =: I_1 + I_2 + I_3,$$

and estimate I_1 and I_2 from below and $|I_3|$ from above.

It is easy to notice that

$$\begin{aligned} I_1 &= \frac{a_{2i-2}+a_{2i-1}}{2}\cdot(s_{2i-1}-s_{2i-2}) + \frac{a_{2i-1}+a_{2i}}{2}\cdot(s_{2i}-s_{2i-1}) + \\ &+ \frac{a_{2i}+a_{2i+1}}{2}\cdot(s_{2i+1}-s_{2i}) = 2^{-k-2}\cdot(a_{2i-2}+2a_{2i-1}+3a_{2i}+2a_{2i+1}). \end{aligned} \tag{13}$$

One can infer from inequalities (8)–(10) that $a_{2i-2} \geq -\dfrac{66}{107}a_{2i-1}$ and $3a_{2i}+2a_{2i+1} =$ $= -3|a_{2i}| + 2\,|a_{2i+1}| \geq -\dfrac{5}{2}\,|a_{2i}| \geq -\dfrac{120}{97}a_{2i-1}$. So, applying (13), we get

$$I_1 \geq 0.02 \cdot 2^{-k} a_{2i-1}. \tag{14}$$

We get analogous to (13)

$$I_2 = ((a_{2i+1}+a_{2i+2}) + (a_{2i+2}+a_{2i+3}))\, 2^{-k-1} = (a_{2i+1}+2a_{2i+2}+a_{2i+3}) 2^{-k-1}. \tag{15}$$

Applying inequalities (8) and (10), we derive

$$a_{2i+1}+2a_{2i+2}+a_{2i+3} > |a_{2i+1}| - 2|a_{2i+2}| > 0.$$

Hence, from (15) we get

$$I_2 > 0. \tag{16}$$

In order to estimate I_3 apply (8). We obtain

$$|I_3| \leq \int_{s_{2i+3}}^{1} |f_n(t)|dt \leq \sum_{j=2i+3}^{n-1} 2^{-k} \cdot \frac{\max(|a_j|,|a_{j+1}|)}{2} \leq 2^{-k-1}|a_{2i}| \sum_{j=3}^{\infty} \left(\frac{2}{7}\right)^j,$$

and then from (9) we get

$$|I_3| \leq 2^{-k-1} \cdot \frac{48}{97}|a_{2i-1}| \cdot \frac{8}{245} < 0.01 \cdot 2^{-k} a_{2i-1}.$$

Therefore, from (14) and (16) we obtain

$$I > I_1 + I_2 - |I_3| > c \cdot 2^{-k} a_{2i-1}.$$

Recall that $\|f_n\|_\infty = a_{2i-1}$ (see (12)) and $1/n \sim 2^{-k}$, therefore applying (11), from the previous estimate we get

$$I > c \cdot \frac{1}{n} \cdot \|f_n\|_\infty \sim n^{-1/2} \sim \|f_n\|_1. \qquad \square$$

Proof of Theorem. We will only consider the case

$$\sup_{m,k} \frac{\lambda_k^m}{\lambda_{k-1}^m \cdot \log \lambda_k^m} = +\infty,$$

since the case $\sup_{m,k} \dfrac{\lambda_{k-1}^m}{\lambda_k^m \cdot \log \lambda_{k-1}^m} = +\infty$ can be done analogously. Since $1/h_m \rightrightarrows 0$ on $[0,1]$, without loss of generality we can assume that $h_1(x) \geq 1$ and

$$\min_{x\in[0,1]} h_m(x) \geq 8m \max_{x\in[0,1]} h_{m-1}(x), \tag{17}$$

hence

$$\lambda_k^m \geq m! \quad \text{for any } m \geq 1,\ 1 \leq k \leq s_m. \tag{18}$$

Moreover, passing to a subsequence of h_m, we can assume that for some sequence k_m

$$\lim_{m\to\infty} \frac{\lambda_{k_m}^m}{\lambda_{k_m-1}^m \cdot \log \lambda_{k_m}^m} = +\infty. \tag{19}$$

Denote $w(m) = \left(\dfrac{\lambda_{k_m}^m}{\lambda_{k_m-1}^m \cdot \log \lambda_{k_m}^m}\right)^{1/2}$. Clearly (see (19))

$$\lim_{m\to\infty} w(m) = +\infty. \tag{20}$$

Let us choose $n_m = 2^k + i$, $1 \le i \le 2^k$, so that

$$n_m \le w(m)\lambda^m_{k_m-1} \cdot \log \lambda^m_{k_m} \le 2n_m, \tag{21}$$

$$s_{n_m,2i-2} = \inf I^m_{k_m}, \tag{22}$$

and choose b_m so that

$$b_m \|f_{n_m}\|_\infty = b_m f_{n_m}(s_{n_m,2i-1}) = \frac{1}{4}\lambda^m_{k_m}. \tag{23}$$

Combining the last equality with the L_p norm estimate (11) and (21), we derive

$$b_m \|f_{n_m}\|_1 \sim \lambda^m_{k_m} \cdot \frac{1}{n_m} \sim w(m) \to \infty. \tag{24}$$

Applying (9), (8) and (18), from (23) we get

$$|b_m f_{n_m}(t)| \le C\lambda^m_{k_m} \left(\frac{2}{7}\right)^{\log \lambda^m_{k_m}} \le \frac{1}{2^m}, \text{ when } |t - s_{n_m,2i-2}| > \log \lambda^m_{k_m} \cdot \frac{2}{n_m}.$$

It follows from (21) that $\log \lambda^m_{k_m} \cdot \frac{2}{n_m} \le \frac{4}{w(m)\lambda^m_{k_m-1}}$, which combined with previous inequality gives the following estimate

$$|b_m f_{n_m}(t)| \le \frac{1}{2^m}, \text{ for } t \in [0, s_{m,-}] \cup [s_{m,+}, 1], \tag{25}$$

where $s_{m,\pm} = s_{n_m,2i-2} \pm \frac{4}{w(m)\lambda^m_{k_m-1}}$. Since (see (2)–(4))

$$1/\lambda^m_{k_m-1} < C_1 |I^m_{k_m-1}| < C_2 |I^m_{k_m}|. \tag{26}$$

We claim that from (20) for big enough m it follows that

$$s_{m,-} \in I^m_{k_m-1} \text{ and } s_{m,+} \in I^m_{k_m}. \tag{27}$$

The series $\sum\limits_{m=1}^{\infty} b_m f_{n_m}(t)$ satisfies all the conditions of Theorem.

Denote $A_m := \left\{t; |b_m f_{n_m}(t)| > \frac{h_m(t)}{4}\right\}$, $B_m := \bigcup\limits_{l=m+1}^{\infty} C_l$, where $C_m := [s_{m,-}, s_{m,+}]$.

It follows from (25), (27) that $A_m \subset C_m \subset I^m_{k_m-1} \cup I^m_{k_m}$, and taking into account (26) and (23), we get

$$A_m \subset I^m_{k_m-1} \text{ and } |A_m| \le \frac{4}{w(m)\lambda^m_{k_m-1}} \le C\frac{|I^m_{k_m-1}|}{w(m)}. \tag{28}$$

Denoting by $S_k(t) = \sum\limits_{i=1}^{k} b_i f_{n_i}(t)$ we have the following estimate from (23) for $k < m$

$$|S_k(t)| \le \sum_{i=1}^{k} |b_i f_{n_i}(t)| \le \sum_{i=1}^{m-1} \max_{t \in [0,1]} h_i(t),$$

hence, applying (17), we get

$$|S_k(t)| \le 2 \max_{t\in[0,1]} h_{m-1}(t) \le \frac{1}{4} \min_{t\in[0,1]} h_m(t). \tag{29}$$

Therefore,

$$|S_m(t)| \leq b_m|f_{n_m}(t)| + |S_{m-1}(t)| \leq \frac{1}{2}h_m(t) \text{ for } t \notin A_m. \tag{30}$$

Combining the last estimate with (29), (25), for any $k \in N$ we get

$$|S_k(t)| \leq \frac{1}{2}h_m(t) + \sum_{l=m+1}^{\infty} \frac{1}{2^l} < h_m(t), \text{ when } t \notin (A_m \cup B_m). \tag{31}$$

Hence,

$$\{t; \sigma^*(t) > h_m(t)\} \subset \{t; \sup_k |S_k(t)| > h_m(t)\} \subset A_m \cup B_m. \tag{32}$$

Let us prove that

$$\lim_{m\to\infty} \int_{A_m\cup B_m} h_m(t)dt = 0. \tag{33}$$

Indeed, from (17) we get $\lambda_i^{l+1} \geq 8\lambda_j^l$, for $l \geq 1$, $1 \leq i, j \leq n_l$, therefore

$$|B_m| \leq \sum_{l=m+1}^{\infty} \frac{8}{w(l)\lambda_{k_l-1}^l} \leq \sum_{l=m+1}^{\infty} \frac{8}{8^{l-m-1}\lambda_{k_{m+1}-1}^{m+1}} \leq \frac{10}{\lambda_{k_{m+1}-1}^{m+1}}. \tag{34}$$

The last inequality and (28) yield

$$\int_{A_m\cup B_m} h_m(t)dt \leq \lambda_{k_m-1}^m |A_m| + \max_{t\in[0,1]} h_m(t)|B_m| \leq \frac{4}{w(m)} + \frac{1}{m}\min_{t\in[0,1]} h_{m+1}(t)\cdot \frac{10}{\lambda_{k_{m+1}-1}^{m+1}} \leq$$

$$\leq \frac{4}{w(m)} + \frac{1}{m}\lambda_{k_{m+1}-1}^{m+1} \cdot \frac{10}{\lambda_{k_{m+1}-1}^{m+1}} = \frac{4}{w(m)} + \frac{10}{m} \to 0.$$

It follows from (32) and (33) that $\displaystyle\int\limits_{\{t;\sigma^*(t)>h_m(t)\}} h_m(t)dt \leq \int\limits_{A_m\cup B_m} h_m(t)dt \to 0$, so the condition (5) is fulfilled.

Now let us prove that the series $\sum\limits_{m=1}^{\infty} b_m f_{n_m}(t)$ converges for a.e. $t \in [0,1]$. Note that for any $m \in N$ this series converges for any $t \notin B_m$. Hence, it also converges on the complement of the set $E = \cap_{m=1}^{\infty} B_m$, and from (34) we have that $|E| = \lim\limits_{m\to\infty} |B_m| = 0$.

It remains to check that $\displaystyle\int_0^1 [f(t)]_{h_m(t)}dt \to \infty$, where $f(t) = \sum_{m=1}^{\infty} b_m f_{n_m}(t)$.

Since $|f(t) - S_m(t)| \leq \sum\limits_{l=m+1}^{\infty} 2^{-l} = 2^{-m}$, for $t \notin B_m$, applying (33), we get

$$\left| \int_0^1 [f(t)]_{h_m(t)}dt - \int_0^1 [S_m(t)]_{h_m(t)}dt \right|$$

$$\leq 2\int_{A_m\cup B_m} h_m(t)dt + \int_{[0,1]\setminus(A_m\cup B_m)} |f(t) - S_m(t)|dt$$

$$\leq 2\int_{A_m\cup B_m} h_m(t)dt + 2^{-m} \to 0, \quad m \to \infty. \tag{35}$$

It follows from (29) that $[S_{m-1}(t)]_{h_m(t)} = S_{m-1}(t)$ and, therefore,

$$\int_0^1 [S_{m-1}(t)]_{h_m(t)}dt = \int_0^1 S_{m-1}(t)dt = 0.$$

Hence from (30) we get

$$\left|\int_0^1 [S_m(t)]_{h_m(t)}dt\right| = \left|\int_0^1 [S_m(t)]_{h_m(t)}dt - \int_0^1 [S_{m-1}(t)]_{h_m(t)}dt\right| \geq$$

$$\left|\int_{[0,1]/A_m} (S_m(t) - S_{m-1}(t))dt\right| - 2\int_{A_m} h_m(t)dt = \left|\int_{[0,1]/A_m} b_m f_{n_m}(t)dt\right| - 2\int_{A_m} h_m(t)dt.$$

Combining this inequality with (35) and (33), we get that in order to complete the proof it remains to prove

$$\lim_{m\to\infty}\left|\int_{[0,1]\setminus A_m} b_m f_{n_m}(t)dt\right| = \infty. \tag{36}$$

Write $\int_{[0,1]\setminus A_m} b_m f_{n_m}(t)dt = \int_{[0,s_{m,-}]} b_m f_{n_m}(t)dt + \int_{[s_{m,-},s_{n_m,2i-2}]\setminus A_m} b_m f_{n_m}(t)dt + \int_{[s_{n_m,2i-2},1]} b_m f_{n_m}(t)dt = I_1 + I_2 + I_3$, and estimate $|I_1|$ and $|I_2|$ from above, while $|I_3|$ from below. It follows from (25) that

$$|I_1| \leq 2^{-m}, \tag{37}$$

and from (26) that

$$|I_2| \leq \int\limits_{[s_{m,-},s_{n_m,2i-2}]\setminus A_m} h_m(t)dt \leq \lambda^m_{k_m-1}\cdot(s_{n_m,2i-2} - s_{m,-}) = \frac{4}{w(m)}. \tag{38}$$

Applying Lemma and (24), we get

$$|I_3| = I_3 \geq c\cdot b_m \|f_{n_m}\|_1 \geq c\cdot w(m). \tag{39}$$

Taking into account (20), from (37)–(39) we get

$$\lim_{m\to\infty}\left|\int\limits_{[0,1]\setminus A_m} b_m f_{n_m}(t)dt\right| \geq \lim_{m\to\infty}(|I_3| - |I_1| - |I_2|) = \infty.$$

This proves equality (36) completing the proof of the Theorem.

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