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M a t h e m a t i c s

UNIQUENESS THEOREMS FOR MULTIPLE FRANKLIN SERIES

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It is proved, that if the square partial sums $\sigma_{q_n}(\mathbf{x})$ of a multiple Franklin series converge in measure to a function f, the ratio $\dfrac{q_{n+1}}{q_n}$ is bounded and the majorant of partial sums satisfies to a necessary condition, then the coefficients of the series are restored by the function f.

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Introduction. The orthonormal Franklin system consists of piecewise linear and continuous functions. This system was constructed by Franklin [1] as the first example of a complete orthonormal system, which is a basis in the space of continuous functions on $[0,1]$. In order to formulate earlier, as well as new results, let's recall some definitions.

Let $n = 2^\mu + \nu$, $\mu \geq 0$, where $1 \leq \nu \leq 2^\mu$. Denote

$$s_{n,i} = \begin{cases} \dfrac{i}{2^{\mu+1}} & \text{for } 0 \leq i \leq 2\nu, \\ \dfrac{i-\nu}{2^\mu} & \text{for } 2\nu < i \leq n. \end{cases} \tag{1}$$

Let S_n denote the space of functions continuous and piecewise linear on $[0,1]$ with nodes $\{s_{n,i}\}_{i=0}^n$, i.e. $f \in S_n$, if $f \in C[0,1]$, is linear on each closed interval $[s_{n,i-1}, s_{n,i}]$, $i = 1,2,\ldots,n$. It is clear, that $\dim S_n = n+1$ and the set $\{s_{n,i}\}_{i=0}^n$ is obtained by adding the point $s_{n,2\nu-1}$ to the set $\{s_{n-1,i}\}_{i=0}^{n-1}$. Therefore, there exists a unique function $f_n \in S_n$, which is orthogonal to S_{n-1}, $\|f_n\|_2 = 1$ and $f_n(s_{n,2\nu-1}) > 0$. Setting $f_0(x) = 1$, $f_1(x) = \sqrt{3}(2x-1)$, $x \in [0,1]$, we obtain the orthonormal system $\{f_n(x)\}_{n=0}^\infty$, which was defined equivalently by Franklin [1].

In this paper we will consider multiple series by Franklin system.

Let d be a natural number. Consider multiple Franklin series

$$\sum_{\mathbf{m} \in \mathbb{N}_0^d} a_{\mathbf{m}} f_{\mathbf{m}}(\mathbf{x}), \tag{2}$$

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where $\mathbf{m}=(m_1,m_2,\ldots,m_d)\in\mathbb{N}_0^d$ is a vector with non-negative integer coordinates, $\mathbf{x}=(x_1,x_2,\ldots,x_d)\in[0,1]^d$ and $f_{\mathbf{m}}(\mathbf{x})=f_{m_1}(x_1)\cdots f_{m_d}(x_d)$.

Denote by $\sigma_n(\mathbf{x})$ the n-th square partial sum of the series (2), i.e.

$$\sigma_n(\mathbf{x})=\sum_{\mathbf{m}:m_i\le n,\ i=1,\ldots,d} a_{\mathbf{m}}f_{\mathbf{m}}(\mathbf{x}), \tag{3}$$

where $\mathbf{m}=(m_1,\ldots,m_d)$.

The following theorems were proved by Gevorkyan and Poghosyan.

***Theorem* A.** [2]. If the sums $\sigma_{2^n}(\mathbf{x})$ converge in measure to an integrable function f and

$$\liminf_{\lambda\to+\infty}\Big(\lambda\cdot\operatorname{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{2^n}(\mathbf{x})|>\lambda\big\}\Big)=0,$$

then the series (2) is the Fourier–Franklin series of f.

***Theorem* B.** [3] If the sums $\sigma_{2^n}(\mathbf{x})$ converge in measure to a function f and

$$\lim_{k\to\infty}\Big(\lambda_k\cdot\operatorname{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{2^n}(\mathbf{x})|>\lambda_k\big\}\Big)=0$$

for some sequence $\lambda_k\to+\infty$, then for any $\mathbf{m}\in\mathbb{N}_0^d$

$$a_{\mathbf{m}}=\lim_{k\to+\infty}\int_{[0,1]^d}[f(\mathbf{x})]_{\lambda_k}f_{\mathbf{m}}(\mathbf{x})d\mathbf{x},$$

where

$$[f(\mathbf{x})]_\lambda=\begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})|\le\lambda,\\ 0, & \text{if } |f(\mathbf{x})|>\lambda.\end{cases}$$

In this paper we will prove, that in the Theorem B instead of the partial sums $\sigma_{2^n}(\mathbf{x})$ one can take square partial sums $\sigma_{q_n}(\mathbf{x})$, where q_n is any increasing sequence of natural numbers, for which the ratio $\dfrac{q_{n+1}}{q_n}$ is bounded. The following theorem holds.

Theorem 1. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\dfrac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function f and there exists a sequence $\lambda_k\to+\infty$ that the following condition holds:

$$\lim_{k\to\infty}\Big(\lambda_k\cdot\operatorname{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{q_n}(\mathbf{x})|>\lambda_k\big\}\Big)=0, \tag{4}$$

then for any $\mathbf{m}\in\mathbb{N}_0^d$

$$a_{\mathbf{m}}=\lim_{k\to+\infty}\int_{[0,1]^d}[f(\mathbf{x})]_{\lambda_k}f_{\mathbf{m}}(\mathbf{x})d\mathbf{x}. \tag{5}$$

Recall, that the function f is called A-integrable on a set G, if $\lim\limits_{\lambda\to+\infty}\lambda\cdot\operatorname{mes}\{x\in G:\ |f(x)|>\lambda\}=0$ and the following limit exists:

$$\lim_{\lambda\to+\infty}\int_G[f(x)]_\lambda dx=:(A)\int_G f(x)dx.$$

Notice that the next two theorems are immediate corollaries of the Theorem 1.

T h e o r e m 2. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\frac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function f and

$$\lim_{\lambda\to\infty}\Big(\lambda\cdot\mathrm{mes}\big\{\mathbf{x}\in[0,1]^d:\ \sup_n|\sigma_{q_n}(\mathbf{x})|>\lambda\big\}\Big)=0,$$

then all functions $f(\mathbf{x})f_{\mathbf{m}}(\mathbf{x})$, $\mathbf{m}\in\mathbb{N}_0^d$, are A-integrable and

$$a_{\mathbf{m}}=(A)\int_{[0,1]^d}f(\mathbf{x})f_{\mathbf{m}}(\mathbf{x})d\mathbf{x},\qquad \mathbf{m}\in\mathbb{N}_0^d.$$

T h e o r e m 3. Let $\{q_n\}$ be an increasing sequence of natural numbers such that the ratio $\frac{q_{n+1}}{q_n}$ is bounded. If the sums $\sigma_{q_n}(\mathbf{x})$ converge in measure to a function $f\in L[0,1]^d$ and for some sequence $\lambda_k\to+\infty$ the condition (4) holds, then (2) is the Fourier–Franklin series of f.

Not that similar questions for series by Franklin system and generlized Franklin system were considered in [4–7].

In [8] for Haar series analogous theorems to Theorems 1–3 are proved.

Similar problems for Vilenkin and generalized Haar systems were considered in [9] and [10], for systems generated by a bounded sequence $\{p_k\}$ and in [11] for general case.

Proof of Theorems. Let $\{q_n\}$ be an increasing sequence of natural numbers and M be a number satisfying the inequality

$$\frac{q_{n+1}}{q_n}\le M\ \text{ for all }\ n\in\mathbb{N}. \tag{6}$$

Denote $S^*(\mathbf{x}):=\sup_n|\sigma_{q_n}(\mathbf{x})|$ and suppose that for the sequence $\lambda\nearrow+\infty$ the following statement holds:

$$\lim_{k\to+\infty}\Big(\lambda_k\cdot\mathrm{mes}\big\{\mathbf{x}\in[0,1]^d:\ S^*(\mathbf{x})>\lambda_k\big\}\Big)=0. \tag{7}$$

Let $\{s_{n,i}\}_{i=0}^n$ be the points given in (1), $s_{n,-1}=0$ and $s_{n,n+1}=1$. For any n and $i\in\{0,1,\dots,n\}$ denote $\delta_i^n:=(s_{n,i-1},s_{n,i+1})$. Let define the function $B_i^n(x)$ as follows. It is linear on intervals $[s_{n,j-1},s_{n,j}]$, $j=1,2,\dots,n$, and

$$B_i^n(s_{n,j})=\begin{cases}1, & \text{if}\quad i=j,\\ 0, & \text{if}\quad i\ne j,\end{cases}\qquad j=0,1,\dots,n.$$

For any natural ν we set $\mathbb{N}_\nu^d:=\{0,1,\dots,q_\nu\}^d$. It is clear that

$$\sigma_{q_\nu}(\mathbf{x})=\sum_{\mathbf{m}\in\mathbb{N}_\nu^d}a_{\mathbf{m}}f_{\mathbf{m}}(\mathbf{x}).$$

For any $\mathbf{j}=(j_1,j_2,\dots,j_d)\in\mathbb{N}_\nu^d$ denote

$$\Delta_{\mathbf{j}}^\nu:=\delta_{j_1}^{q_\nu}\times\delta_{j_2}^{q_\nu}\times\cdots\times\delta_{j_d}^{q_\nu}, \tag{8}$$
$$F_{\mathbf{j}}^\nu(\mathbf{x}):=B_{j_1}^{q_\nu}(x_1)\cdot B_{j_2}^{q_\nu}(x_2)\cdot\ldots\cdot B_{j_d}^{q_\nu}(x_d).$$

Obviously $\mathrm{supp}(F_{\mathbf{j}}^\nu)=\overline{\Delta_{\mathbf{j}}^\nu}$ and

$$\left(\frac{1}{2q_\nu}\right)^d\le\mathrm{mes}(\Delta_{\mathbf{j}}^\nu)\le\left(\frac{4}{q_\nu}\right)^d. \tag{9}$$

It follows from the definition of functions B_i^n, that $\sum_{i=0}^{n} B_i^n(x)=1$ for all $x\in[0,1]$, therefore,

$$\sum_{\mathbf{j}\in\mathbb{N}_\nu^d} F_{\mathbf{j}}^{\nu}(\mathbf{x})=1 \quad \text{for all} \quad \mathbf{x}\in[0,1]^d.$$

Let us notice that

$$\int_{[0,1]^d} F_{\mathbf{j}}^{\nu}(\mathbf{x})d\mathbf{x}=\int_{\Delta_{\mathbf{j}}^{\nu}} F_{\mathbf{j}}^{\nu}(\mathbf{x})d\mathbf{x}=\prod_{i=1}^{d}\int_{\delta_{j_i}^{q_\nu}} F_{j_i}^{q_\nu}(x_i)dx_i=\frac{\operatorname{mes}(\Delta_{\mathbf{j}}^{\nu})}{2^d}.$$

Therefore, by denoting

$$M_{\mathbf{j}}^{\nu}(\mathbf{x}):=\frac{2^d}{\operatorname{mes}(\Delta_{\mathbf{j}}^{\nu})}F_{\mathbf{j}}^{\nu}(\mathbf{x}),$$

we obtain (in view of (9)) that

$$\int_{[0,1]^d} M_{\mathbf{j}}^{\nu}(\mathbf{x})d\mathbf{x}=1 \quad \text{and} \quad |M_{\mathbf{j}}^{\nu}(\mathbf{x})|\le(4q_\nu)^d, \quad \nu\in\mathbb{N},\ \mathbf{j}\in\mathbb{N}_\nu^d. \tag{10}$$

It is clear that the functions $\{M_{\mathbf{j}}^{\nu}\}_{\mathbf{j}\in\mathbb{N}_\nu^d}$ are basis in the space

$$S_\nu:=\left\{\sum_{\mathbf{m}\in\mathbb{N}_\nu^d} a_{\mathbf{m}}f_{\mathbf{m}}(\mathbf{x}):\ a_{\mathbf{m}}\in\mathbb{R}\right\}.$$

The following lemmas were proved in [2].

Lemma 1. Let F be a function, which is defined on $\Delta=[a_1,b_1]\times\cdots\times[a_d,b_d]$, $d\in\mathbb{N}$, and is linear with respect to each variable. If $L=\max_{t\in\Delta}|F(t)|$, then

$$\operatorname{mes}\left\{t\in\Delta:\ |F(t)|\ge\frac{L}{2^d}\right\}\ge\frac{\operatorname{mes}(\Delta)}{3^d}.$$

Lemma 2. For any $M_{\mathbf{j}_0}^{\nu_0}$ and $\nu>\nu_0$ there exist numbers $\alpha_{\mathbf{j}}$ such that

$$M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x})=\sum_{\mathbf{j}\in\mathbb{N}_\nu^d}\alpha_{\mathbf{j}}M_{\mathbf{j}}^{\nu}(\mathbf{x}),$$

where

$$\sum_{\mathbf{j}\in\mathbb{N}_\nu^d}\alpha_{\mathbf{j}}=1, \quad \alpha_{\mathbf{j}}\ge0 \quad \text{and} \quad \alpha_{\mathbf{j}}=0 \quad \text{if} \quad \Delta_{\mathbf{j}}^{\nu}\not\subset\Delta_{\mathbf{j}_0}^{\nu_0}.$$

Although the Lemma 2 in [2] was proved for $q_\nu=2^\nu$, obviously the same proof is true in general case, also.

Now suppose that the statements (6) and (7) hold and the sums $\sigma_{q_\nu}(\mathbf{x})$ converge in measure to a function f. First let's prove that for any $\mathbf{m}=(m_1,m_2,\ldots,m_d)\in\mathbb{N}_0^d$ and $\mathbf{j}_0\in\mathbb{N}_{\nu_0}^d$, for which $\max_{1\le i\le d}\{m_i\}\le q_{\nu_0}$, the following statement is true:

$$\int_{[0,1]^d}\sigma_{q_{\nu_0}}(\mathbf{x})M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x})d\mathbf{x}=\lim_{k\to+\infty}\int_{[0,1]^d}[f(\mathbf{x})]_{\lambda_k}M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x})d\mathbf{x}. \tag{11}$$

For any $k\in\mathbb{N}$ denote

$$E_k:=\left\{\mathbf{x}\in\operatorname{supp}(M_{\mathbf{j}_0}^{\nu_0})=\overline{\Delta_{\mathbf{j}_0}^{\nu_0}}:\ S^*(\mathbf{x})>\lambda_k\right\}.$$

Let ε be a positive number. In view of (7), one can take the natural number k_0 such that the following inequalities hold:

$$(4^7 q_{\nu_0} M)^d \lambda_k \cdot \text{mes}(E_k) < \varepsilon, \quad \text{when} \quad k \geq k_0, \tag{12}$$

$$\text{mes}(E_k) < (4^4 M)^{-d} \text{mes}(\Delta_{\mathbf{j}_0}^{\nu_0}), \quad \text{when} \quad k \geq k_0. \tag{13}$$

Suppose $\nu \geq \nu_0$. We set

$$\Omega_\nu := \Big\{ A : \ A = (s_{q_\nu, j_1 - 1}, s_{q_\nu, j_1}) \times \ldots \times (s_{q_\nu, j_d - 1}, s_{q_\nu, j_d}), \ A \subset \Delta_{\mathbf{j}_0}^{\nu_0} \Big\}. \tag{14}$$

Obviously

$$\left(\frac{1}{2q_\nu} \right)^d \leq \text{mes}(A) \leq \left(\frac{2}{q_\nu} \right)^d \quad \text{for all} \quad A \in \Omega_\nu. \tag{15}$$

Notice, that if for some $A \in \Omega_\nu, \ \nu \geq \nu_0$, the inequality

$$\text{mes}(E_k \cap A) \leq 2^{-2d} \text{mes}(A) \tag{16}$$

holds, then

$$\left| \sigma_{q_\nu}(\mathbf{x}) \right| \leq 2^d \lambda_k \quad \text{for all} \quad \mathbf{x} \in A. \tag{17}$$

Let suppose that $A \in \Omega_\nu$ and for some point $\mathbf{x}_0 \in A$ the inequality (17) does not hold, i.e. $\left| \sigma_{q_\nu}(\mathbf{x}_0) \right| > 2^d \lambda_k$. Since $\sigma_{q_\nu}(\mathbf{x})$ is linear with respect to each variable on the set A, according to the Lemma 1 we obtain that

$$\text{mes}\big\{ \mathbf{x} \in A : \ \left| \sigma_{q_\nu}(\mathbf{x}) \right| \geq \lambda_k \big\} \geq 3^{-d} \text{mes}(A),$$

which contradicts (16).

According to (8), (13) and (14), we obtain that

$$\text{mes}(E_k \cap A) \leq \big(4^4 M\big)^{-d} \text{mes}\big(\Delta_{\mathbf{j}_0}^{\nu_0}\big) < \big(4^3 M\big)^{-d} \text{mes}(A), \quad \text{when} \quad A \in \Omega_{\nu_0}. \tag{18}$$

Now let's define by induction the families Ω_ν^1 and Ω_ν^2, $\nu \geq \nu_0$. If $\nu = \nu_0$, then we set

$$\Omega_{\nu_0}^1 := \{ A \in \Omega_{\nu_0} : \ \text{mes}(E_k \cap A) > (4^3 M)^{-d} \cdot \text{mes}(A) \}, \quad Q_{\nu_0} := \bigcup_{A \in \Omega_{\nu_0}^1} A,$$

and

$$\Omega_{\nu_0}^2 := \{ A \in \Omega_{\nu_0} : \ A \not\subset Q_{\nu_0} \}, \quad P_{\nu_0} := \bigcup_{A \in \Omega_{\nu_0}^2} A.$$

From (18) we have, that $Q_{\nu_0} = \emptyset$ and the closure of P_{ν_0} is the $\text{supp}(M_{\mathbf{j}_0}^{\nu_0})$, i.e. $\overline{P_{\nu_0}} = \overline{\Delta_{\mathbf{j}_0}^{\nu_0}}$. Now suppose we have defined the sets Ω_n^1, Ω_n^2, Q_n and P_n for all $n < \nu$. Let's denote

$$\Omega_\nu^1 := \Big\{ A \in \Omega_\nu : \ \text{mes}(E_k \cap A) > (4^3 M)^{-d} \cdot \text{mes}(A) \ \text{ and } \ A \not\subset \bigcup_{n < \nu} Q_n \Big\}, \tag{19}$$

$$Q_\nu := \bigcup_{A \in \Omega_\nu^1} A,$$

$$\Omega_\nu^2 := \Big\{ A \in \Omega_\nu : \ A \not\subset \bigcup_{n \leq \nu} Q_n \Big\}, \quad P_\nu := \bigcup_{A \in \Omega_\nu^2} A.$$

Thus we have defined the families Ω_ν^1, Ω_ν^2 and the sets P_ν, Q_ν ($\nu \geq \nu_0$), satisfying to the following conditions: $\Omega_\nu^1 \subset \Omega_\nu$, $\Omega_\nu^2 \subset \Omega_\nu$,

$$\operatorname{supp}(M_{\mathbf{j}_0}^{\nu_0}) = \Delta_{\mathbf{j}_0}^{\nu_0} = \overline{P_\nu} \bigcup \left(\bigcup_{n \leq \nu} \overline{Q_n} \right), \quad P_\nu \bigcap \left(\bigcup_{n \leq \nu} Q_n \right) = \emptyset, \tag{20}$$

$$Q_\nu \cap Q_n = \emptyset, \quad \text{if} \quad \nu \neq n. \tag{21}$$

It is seen from (21) and (19), that

$$\operatorname{mes}\left(\bigcup_{n \leq \nu} Q_n \right) < (4^3 M)^d \operatorname{mes}(E_k) \ \text{ for any } \quad \nu \geq \nu_0. \tag{22}$$

For any $\nu > \nu_0$ denote

$$J_\nu := \{\mathbf{j} \in \mathbb{N}_\nu^d : \ \Delta_{\mathbf{j}}^\nu \cap Q_\nu \neq \emptyset, \ \ \Delta_{\mathbf{j}}^\nu \subset \overline{P_{\nu-1}}\}. \tag{23}$$

Note that for any $\mathbf{j} \in J_\nu$ and for all $B \in \Omega_\nu$, which are subset of $\Delta_{\mathbf{j}}^\nu$, the inequality

$$\operatorname{mes}(E_k \cap B) < 4^{-d} \operatorname{mes}(B) \tag{24}$$

holds. Suppose there exists a parallelepiped $B \in \Omega_\nu$ such that $B \subset \Delta_{\mathbf{j}}^\nu$, but the inequality (24) does not hold. Denote by D that set from $\Omega_{\nu-1}$, which contains B. Using (6) and (15), we get that

$$\operatorname{mes}(B) \geq \left(\frac{1}{2q_\nu} \right)^d \geq \left(\frac{1}{2Mq_{\nu-1}} \right)^d \geq \left(\frac{1}{4M} \right)^d \operatorname{mes}(D).$$

Therefore,

$$\operatorname{mes}(E_k \cap D) \geq \operatorname{mes}(E_k \cap B) \geq 4^{-d} \operatorname{mes}(B) \geq (4^2 M)^{-d} \operatorname{mes}(D),$$

which means that $B \subset D \subset \bigcup_{n<\nu} Q_n$, moreover $\Delta_{\mathbf{j}}^\nu \bigcap \left(\bigcup_{n<\nu} Q_n \right) \neq \emptyset$ (see (19)). But this contradicts to (23) and (20). Thus, if $\mathbf{j} \in J_\nu$, then for all $B \in \Omega_\nu$ with $B \in \Delta_{\mathbf{j}}^\nu$ the inequality (24) is true, therefore,

$$\operatorname{mes}(E_k \cap \Delta_{\mathbf{j}}^\nu) < 4^{-d} \operatorname{mes}(\Delta_{\mathbf{j}}^\nu).$$

Using the last inequality and according to (16) and (17), we get

$$|\sigma_{q_\nu}(\mathbf{x})| \leq 2^d \lambda_k, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{j}}^\nu, \quad \mathbf{j} \in J_\nu. \tag{25}$$

Similarly we obtain (according to definition of P_ν and (19)), that if $\Delta_{\mathbf{j}}^\nu \subset P_\nu$, then $\operatorname{mes}(E_k \cap \Delta_{\mathbf{j}}^\nu) \leq (4^3 M)^{-d} \operatorname{mes}(\Delta_{\mathbf{j}}^\nu)$ and, therefore,

$$|\sigma_{q_\nu}(\mathbf{x})| \leq 2^d \lambda_k, \quad \text{if} \quad \mathbf{x} \in \Delta_{\mathbf{j}}^\nu \subset P_\nu. \tag{26}$$

Now let's define by induction different expansions φ_n for $M_{\mathbf{j}_0}^{\nu_0}$, satisfying conditions:

$$M_{\mathbf{j}_0}^{\nu_0} = \varphi_n = \sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n M_{\mathbf{j}}^\nu + \sum_{\mathbf{j}:\Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n M_{\mathbf{j}}^n, \tag{27}$$

$$\sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n + \sum_{\mathbf{j}:\Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n = 1, \quad \alpha_{\nu,\mathbf{j}}^n \geq 0, \quad \alpha_{\mathbf{j}}^n \geq 0. \tag{28}$$

Set $\varphi_{\nu_0} := M_{\mathbf{j}_0}^{\nu_0}$. It is clear that φ_{ν_0} satisfies both (27) and (28).

Suppose we have defined expansions $\varphi_{\nu_0},\dots,\varphi_n$, satisfying (27) and (28). According to Lemma 2, for any $\Delta_{\mathbf{j}}^n \subset P_n$ we have

$$M_{\mathbf{j}}^n = \sum_{\mathbf{i}:\Delta_{\mathbf{i}}^{n+1}\subset\Delta_{\mathbf{j}}^n} \beta_{\mathbf{i}} M_{\mathbf{i}}^{n+1}, \quad \text{where} \quad \beta_{\mathbf{i}} \geq 0. \tag{29}$$

Note, that if $\Delta_{\mathbf{j}}^n \subset P_n$ and (29) holds, then either $\Delta_{\mathbf{i}}^{n+1} \cap Q_{n+1} \neq \emptyset$ and, therefore, $\mathbf{i} \in J_{n+1}$ or $\Delta_{\mathbf{i}}^{n+1} \subset P_{n+1}$. Therefore, inserting the expressions (29) in (27) and grouping similar terms, we obtain

$$M_{\mathbf{j}_0}^{\nu_0} = \varphi_{n+1} = \sum_{\nu\leq n+1}\sum_{\mathbf{j}\in J_\nu} \alpha_{\nu,\mathbf{j}}^{n+1} M_{\mathbf{j}}^{\nu} + \sum_{\mathbf{j}:\Delta_{\mathbf{j}}^{n+1}\subset P_{n+1}} \alpha_{\mathbf{j}}^{n+1} M_{\mathbf{j}}^{n+1}. \tag{30}$$

It is obvious, that all coefficients in (30) are nonnegative. Since the integrals of all functions $M_{\mathbf{j}}^{\nu}$ are 1 (see (10)), from (30) we get that

$$\sum_{\nu\leq n+1}\sum_{\mathbf{j}\in J_\nu} \alpha_{\nu,\mathbf{j}}^{n+1} + \sum_{\mathbf{j}:\Delta_{\mathbf{j}}^{n+1}\subset P_{n+1}} \alpha_{\mathbf{j}}^{n+1} = 1.$$

So we have proved, that for any $n \geq \nu_0$ the expansion (27) with coefficients (28) is possible.

According to the definition of J_ν and sets Q_ν, we obtain that

$$\operatorname{mes}\left(\bigcup_{\mathbf{j}\in J_\nu} \Delta_{\mathbf{j}}^{\nu}\right) \leq 4^d \operatorname{mes}(Q_\nu).$$

Therefore, using the inequality (22) and (21), for the measure of the set $D_n := \bigcup_{\nu\leq n}\bigcup_{\mathbf{j}\in J_\nu} \Delta_{\mathbf{j}}^{\nu}$ we get that

$$\operatorname{mes}(D_n) \leq 4^d \operatorname{mes}\left(\bigcup_{\nu\leq n} Q_\nu\right) \leq (4^4 M)^d \operatorname{mes}(E_k). \tag{31}$$

According to (10), (27), (28) and (31), we obtain that for any $n \geq \nu_0$

$$\sum_{\nu\leq n}\sum_{\mathbf{j}\in J_\nu} \alpha_{\nu,\mathbf{j}}^{n} = \sum_{\nu\leq n}\sum_{\mathbf{j}\in J_\nu} \alpha_{\nu,\mathbf{j}}^{n} \int_{D_n} M_{\mathbf{j}}^{\nu}(\mathbf{x})d\mathbf{x} \leq \int_{D_n} M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x})d\mathbf{x} \leq (4^5 M q_{\nu_0})^d \operatorname{mes}(E_k). \tag{32}$$

Suppose we are given a number $\nu \geq \nu_0$ and $\mathbf{p} = (p_1, p_2, \dots, p_d) \in \mathbb{N}_0^d$ such that $\max_i\{p_i\} > q_\nu$. Then, according to the definition of functions $f_{\mathbf{p}}$ and $M_{\mathbf{j}}^{\nu}$, we get

$$(f_{\mathbf{p}}, M_{\mathbf{j}}^{\nu}) := \int_{[0,1]^d} f_{\mathbf{p}}(\mathbf{x}) M_{\mathbf{j}}^{\nu}(\mathbf{x})d\mathbf{x} = 0 \quad \text{for any} \quad \mathbf{j} \in \mathbb{N}_\nu^d.$$

Therefore, for any $n \geq \nu$ and for all $\mathbf{j} \in \mathbb{N}_\nu^d$ one can write

$$(\sigma_{q_n}, M_{\mathbf{j}}^{\nu}) = \sum_{\mathbf{p}\in\mathbb{N}_n^d} a_{\mathbf{p}}(f_{\mathbf{p}}, M_{\mathbf{j}}^{\nu}) = \sum_{\mathbf{p}\in\mathbb{N}_\nu^d} a_{\mathbf{p}}(f_{\mathbf{p}}, M_{\mathbf{j}}^{\nu}) = (\sigma_{q_\nu}, M_{\mathbf{j}}^{\nu}). \tag{33}$$

It is easily seen (see Eqs. (27) and (33)) that for any $n \geq \nu_0$

$$\left|\int_{[0,1]^d} \sigma_{q_{\nu_0}}(\mathbf{x}) M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x}) d\mathbf{x} - \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} M_{\mathbf{j}_0}^{\nu_0}(\mathbf{x}) d\mathbf{x}\right| = \left|\left(\sigma_{q_n} - [f]_{\lambda_k}, M_{\mathbf{j}_0}^{\nu_0}\right)\right| =$$

$$= \left|\left(\sigma_{q_n} - [f]_{\lambda_k}, \sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n M_{\mathbf{j}}^\nu + \sum_{\mathbf{j}:\Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n M_{\mathbf{j}}^n\right)\right| \leq \tag{34}$$

$$\leq \left|\sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n \left(\sigma_{q_\nu} - [f]_{\lambda_k}, M_{\mathbf{j}}^\nu\right)\right| + \left|\sum_{\mathbf{j}:\Delta_{\mathbf{j}}^n \subset P_n} \alpha_{\mathbf{j}}^n \left(\sigma_{q_n} - [f]_{\lambda_k} M_{\mathbf{j}}^n\right)\right| =: I_1 + I_2.$$

Using (25), (10), (32) and (12), for I_1 we will have the inequality

$$I_1 \leq \sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n | \left(\sigma_{q_\nu} - [f]_{\lambda_k}, M_{\mathbf{j}}^\nu\right) | \leq (2^d \lambda_k + \lambda_k) \sum_{\nu \leq n} \sum_{\mathbf{j} \in J_\nu} \alpha_{\nu,\mathbf{j}}^n \leq$$

$$\leq (4^6 M q_{\nu_0})^d \lambda_k \mathrm{mes}(E_k) < 4^{-d} \varepsilon, \quad \text{for any} \quad n \geq \nu_0. \tag{35}$$

Denote $H_n := \bigcup_{\mathbf{j}:\Delta_{\mathbf{j}}^n \subset P_n} \Delta_{\mathbf{j}}^n, \qquad T_k := \{\mathbf{x} \in \Delta_{\mathbf{j}_0}^{\nu_0} : \ |f(\mathbf{x})| > \lambda_k\}.$
It is clear (see also (12)) that

$$\mathrm{mes}(T_k) \leq \mathrm{mes}(E_k) \leq (4^7 M q_{\nu_0})^{-d} \lambda_k^{-1} \varepsilon.$$

According to (10) and (27), we obtain

$$\begin{aligned} I_2 &\leq (4q_{\nu_0})^d \int_{H_n} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} = \\ &= (4q_{\nu_0})^d \int_{H_n \cap T_k} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} + \\ &+ (4q_{\nu_0})^d \textstyle\int_{H_n \setminus T_k} |\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}| d\mathbf{x} := I_3 + I_4. \end{aligned} \tag{36}$$

Using (26) and (12), we can estimate I_3 as follows:

$$I_3 \leq (4q_{\nu_0})^d (2^d \lambda_k + \lambda_k)(4^7 M q_{\nu_0})^{-d} \lambda_k^{-1} \varepsilon < 4^{-d} \varepsilon. \tag{37}$$

Since $\sigma_{q_n}(\mathbf{x}) - [f(\mathbf{x})]_{\lambda_k}$ on the set $H_n \setminus T_k$ converges in measure to 0, as $n \to \infty$, and is bounded, then for sufficiently large n we get that $I_4 < \dfrac{\varepsilon}{4}$. Therefore, according to (34)–(37), we obtain (11).

Now let's prove that for any $\mathbf{m} \in \mathbb{N}_0^d$ the coefficient $a_{\mathbf{m}}$ can be found by (5). Assume $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d$. First let's fix a number ν satisfying $\max_{1 \leq i \leq d} m_i \leq q_\nu$. Since $f_{\mathbf{m}} \in S_{q_\nu}$ and the system of functions $\{M_{\mathbf{j}}^\nu\}_{\mathbf{j} \in \mathbb{N}_\nu^d}$ is a basis in the space S_{q_ν}, then one can find numbers $\beta_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{N}_\nu^d$, such that

$$f_{\mathbf{m}}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{N}_\nu^d} \beta_{\mathbf{j}} M_{\mathbf{j}}^\nu(\mathbf{x}). \tag{38}$$

Using (3), (38) and (11), we get that

$$a_{\mathbf{m}} = (\sigma_{q_\nu}, f_{\mathbf{m}}) = \sum_{\mathbf{j}\in\mathbb{N}_\nu^d} \beta_{\mathbf{j}}(\sigma_{q_\nu}, M_{\mathbf{j}}^\nu) = \sum_{\mathbf{j}\in\mathbb{N}_\nu^d} \beta_{\mathbf{j}} \lim_{k\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} M_{\mathbf{j}}^\nu(\mathbf{x}) d\mathbf{x} =$$

$$= \lim_{k\to\infty} \int_{[0,1]^d} [f(\mathbf{x})]_{\lambda_k} f_{\mathbf{m}}(\mathbf{x}) d\mathbf{x},$$

which proves the Theorem 1.

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