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Mathematics

A REPRESENTATION FOR THE SUPPORT FUNCTION OF A CONVEX BODY

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In this paper a formula for a translation invariant measure of planes intersecting a n-dimensional convex body in terms of curvatures of 2-dimensional projections of the body was found. The paper also gives a new simple proof of the representation for the support function of an origin symmetric 3-dimensional convex body, which was obtained by means of a stochastic approximation of the convex body.

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Introduction. We denote by $\mathbb{R}^n$ $(n \geq 2)$ the Euclidean n-dimensional space. Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and let λ_{n-1} be the spherical Lebesgue measure on $\mathbf{S}^{n-1}$ $\big(\lambda_k(\mathbf{S}^k) = \sigma_k\big)$. Denote by $\mathbf{S}_\omega \subset \mathbf{S}^{n-1}$ the greatest $(n-2)$-dimensional circle with pole at $\omega \in \mathbf{S}^{n-1}$. The class of the origin symmetric convex bodies (nonempty compact convex sets) $\mathbf{B}$ in $\mathbb{R}^n$ we denote by $\mathcal{B}^n_o$ (so called the *centered* bodies).

The most useful analytic description of compact convex sets is given by the support function (see [1]). The support function $H : \mathbb{R}^n \to (-\infty, \infty]$ of a body $\mathbf{B}$ is defined by

$$H(x) = \sup_{y \in \mathbf{B}} \langle y, x \rangle, \ \ x \in \mathbb{R}^n.$$

Here and below $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. The support function of $\mathbf{B}$ is positively homogeneous and convex. Below, we consider the support function $H(\cdot)$ of a convex body as a function defined on the unit sphere $\mathbf{S}^{n-1}$ (because of the positive homogeneity of $H(\cdot)$, the values on $\mathbf{S}^{n-1}$ determine $H(\cdot)$ completely).

It is well known that any convex body $\mathbf{B}$ is uniquely determined by its support function [1].

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A convex body $\mathbf{B}$ is k-smooth, if its support function $H \in \mathbf{C}^k(\mathbf{S}^{n-1})$, where $\mathbf{C}^k(\mathbf{S}^{n-1})$ denotes the space of k times continuously differentiable functions defined on $\mathbf{S}^{n-1}$.

It is known (see [2, 3]) that *the support function $H(\cdot)$* of a sufficiently smooth origin symmetric convex body $\mathbf{B} \in \mathcal{B}_o^n$ has the following representation:

$$H(\xi) = \int_{\mathbf{S}^{n-1}} |\,(\xi,\Omega)\,|\, h(\Omega)\,\lambda_{n-1}(d\Omega), \;\; \xi \in \mathbf{S}^{n-1}, \tag{1}$$

with *an even continuous function $h(\cdot)$* (not necessarily positive) defined on $\mathbf{S}^{n-1}$. Note that h is unique. Such bodies (whose support functions have the integral representation (1) with a signed even measure) are called *generalized zonoids*. In the case when h is a positive function on $\mathbf{S}^{n-1}$ the centrally symmetric convex body $\mathbf{B}$ is a zonoid.

In this article a formula for a translation invariant measure of planes intersecting an n-dimensional convex body in terms of curvatures of 2-dimensional projections of the body was found. The paper also gives a new simple prove of the representation for the support function of an origin symmetric 3-dimensional convex body (see Theorem 3), which was obtained by means of a stochastic approximation of the convex body (see [4]).

A Representation for the Translation Invariant Measure. Let $\mathbf{B}$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of the boundary $\partial\mathbf{B}$: $k_1(\omega)\cdots k_{n-1}(\omega) > 0$, where $k_1(\omega),...,k_{n-1}(\omega)$ signify the principal curvature of $\partial\mathbf{B}$ at the point with outer normal direction $\omega \in \mathrm{S}^{n-1}$.

For two different directions $\omega,\xi \in \mathrm{S}^{n-1}$, $\omega \neq \xi$, we denote by $B(\omega,\xi)$ the projection of $\mathbf{B}$ onto the 2-dimensional plane $e(\omega,\xi)$ containing the origin and the directions ω and ξ. Let $R(\omega,\xi)$ be the curvature radius of $\partial B(\omega,\xi)$ at a point, whose outer normal direction is ω, which is said to be the 2-dimensional projection curvature radius of the body. Denot by $\widehat{(\omega,\xi)}$ the angle between ξ and ω. Since $R(\omega,\xi_1) = R(\omega,\xi_2)$, where $\omega,\xi_1,\xi_2 \in \mathrm{S}^{n-1}$ are linearly dependent vectors, we assume where necessary that ξ is orthogonal to ω.

Let μ be a translation invariant measure in the space $\mathbf{E}^n$ of hyperplanes in $\mathbb{R}^n$. It is known that the translation invariant measure μ can be decomposed, that is there exists a finite even measure m_{n-1} on S^{n-1} such that

$$d\mu = dp \cdot m_{n-1}(d\xi),$$

where (p,ξ) is the usual parametrization of a hyperplane e, i.e. p is the distance from the origin O to e, $\xi \in \mathrm{S}^{n-1}$ is the direction normal to e (see [5]). m_{n-1} is called the rose of directions of μ. We denote by $[\mathrm{B}]$ the set of hyperplanes intersecting $\mathbf{B}$.

Note that in the case when the translation invariant measure μ is concentrated on the bundle of parallel hyperplanes orthogonal to $\xi \in \mathrm{S}^{n-1}$, we will have $\mu[\mathrm{B}] = 2H(\xi)$ for $\mathbf{B} \in \mathcal{B}_o^n$.

Theorem 1. Let μ be a translation invariant measure in $\mathbf{E}^n$ with the rose of directions m_{n-1}. For any 2 smooth convex body $\mathrm{B} \in \mathcal{B}_o^n$ we have the following representation:

$$\mu([\mathrm{B}]) = \frac{1}{2\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \frac{R(\omega,\xi)}{\sin^{n-3}\widehat{(\omega,\xi)}}\,\lambda_{n-1}(d\omega) m_{n-1}(d\xi). \tag{2}$$

Proof. We have

$$\mu([\mathrm{B}]) = \int_{[\mathrm{B}]} dp\, m_{n-1}(d\xi) = \int_{\mathbf{S}^{n-1}} H(\xi)\, m_{n-1}(d\xi). \tag{3}$$

Now we are going to find the representation for the support function of an origin symmetric n-dimensional convex body $\mathrm{B} \in \mathcal{B}_o^n$.

Let $u \in \mathrm{S}_\xi$ be a direction perpendicular to $\xi \in \mathrm{S}^{n-1}$. Approximating $\mathrm{B}(u,\xi) \subset e(u,\xi)$ from inside by polygons that have their vertices on $\partial\mathrm{B}(u,\xi)$. Let denote by a_i the sides of the approximation polygon, by ν_i the direction normal to a_i within $e(u,\xi)$ (let also denote by ν_i the angle between the normal to a_i and ξ). Let $H_{\mathrm{B}(u,\xi)}$ be the support function of $\mathrm{B}(u,\xi)$ in the plane $e(u,\xi)$. We have

$$4H(\xi) = 4H_{\mathrm{B}(u,\xi)}(\xi) = \lim \sum_i |a_i| \sin(\xi,\nu_i) =$$

$$= \lim \sum_i R_u(\nu_i)|\nu_{i+1} - \nu_i| \sin(\xi,\nu_i) = 2\int_0^\pi R_u(\nu)\sin\nu\, d\nu, \tag{4}$$

where $R_u(\nu)$ is radius of the curvature of $\mathrm{B}(u,\xi)$ at the point with normal ν. Integrating both sides of Eq. (4) in $\lambda_{n-2}(du)$ over $\mathbf{S}_\xi$, and using standard formula $\lambda_{n-1}(d\omega) = \sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(du)$, where $\omega = (\nu,u)$, we obtain (see also [6])

$$2\sigma_{n-2}H(\xi) = \int_{\mathbf{S}_\xi}\int_0^\pi R_u(\nu)\sin\nu\, d\nu\, \lambda_{n-2}(du) =$$

$$= \int_{\mathbf{S}_\xi}\int_0^\pi \frac{R_u(\nu)}{\sin^{n-3}\nu}\sin^{n-2}\nu\, d\nu\, \lambda_{n-2}(du) = \int_{\mathbf{S}^{n-1}} \frac{R(\omega,\xi)}{\sin^{n-3}(\widehat{\omega,\xi})}\lambda_{n-1}(d\omega). \tag{5}$$

Substituting (5) written for $H(\xi)$ into (3), we obtain (2). □

Note that replacing $2H(\cdot)$ by the width function $W(\cdot)$ in Eq. (4), we get a formula for the width function for any convex body (not only centrally symmetric). Hence, the representation (2) is valid for any convex body.

Using (2) one can obtain a representation for $\mu[\mathrm{B}]$ in terms of the principal radii of curvatures of the boundary of **B**. Further, assuming that $s(\omega)$ is the point on $\partial\mathbf{B}$, which outer normal is ω, we will get that $R_i(\omega)$ is the i-th principal radii of curvature $(i = 1,...,n-1)$ of $\partial\mathrm{B}$ at $s(\omega)$.

Theorem 2. Let μ be a translation invariant measure in $\mathbf{E}^n$ with the rose of directions m_{n-1}. For any 2 smooth convex body $\mathrm{B} \in \mathcal{B}_o^n$ we have the following representation:

$$\mu([\mathrm{B}]) = \frac{1}{2\sigma_{n-2}}\int_{\mathbf{S}^{n-1}}\left[\sum_{i=1}^{n-1} R_i(\omega)\int_{\mathbf{S}^{n-1}} \frac{\cos^2\varphi_i}{\sin^{n-3}(\widehat{\omega,\xi})} m_{n-1}(d\xi)\right]\lambda_{n-1}(d\omega), \tag{6}$$

where φ_i is the angle between the i-th principal direction at $s(\omega) \in \partial\mathrm{B}$ and the projection of ξ onto the tangent plane of $\partial\mathrm{B}$ at $s(\omega)$.

Proof. For any $\omega \in \mathbf{S}^{n-1}$ and $\xi \in \mathbf{S}_\omega^{n-2}$ the following formula for the radius of the projection curvature of **B** is valid (see [7]):

$$R(\omega,\xi) = \sum_{i=1}^{n-1} R_i(\omega)\cos^2\varphi_i. \tag{7}$$

Substituting (7) into (2) and applying Fubini's theorem, we obtain (6). □

Note, that the representation (6) first was found by Panina [8] using another method.

If $\mu = \mu_{inv}$ is an invariant measure in the space $\mathbf{E}^n$, i.e. $\mu_{inv}(de) = dp \times \lambda_{n-1}(d\xi)$ (see [7]), so we have

Corollary. For any 2 smooth convex body B we have the following representation

$$\mu_{inv}([\mathrm{B}]) = \frac{1}{n-1} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^{n-1} R_i(\omega) \lambda_{n-1}(d\omega). \tag{8}$$

Indeed, let us assume that ξ has usual spherical coordinates (τ, u) $\big($where $\tau \in (0, \pi)$, $u \in \mathbf{S}^{n-2}_{\omega}\big)$ with respect ω as the North Pole.

Substituting $\lambda_{n-1}(d\xi) = \sin^{n-2}\tau \, d\tau \, \lambda_{n-2}(du)$ into (6), we obtain

$$\mu([\mathrm{B}]) = \frac{a_n}{\sigma_{n-2}} \int_{\mathbf{S}^{n-1}} \sum_{i=1}^{n-1} R_i(\omega) \, \lambda_{n-1}(d\omega),$$

where

$$\frac{a_n}{\sigma_{n-2}} = \frac{\sigma_{n-3} \displaystyle\int_0^{\pi} \cos^2 \nu \, \sin^{n-3} \nu \, d\nu}{\sigma_{n-3} \displaystyle\int_0^{\pi} \sin^{n-3} \nu \, d\nu} = \frac{1}{n-1}.$$

Note that for $n = 3$ (see Eq. (8)) coincides with the Minkowski's formula in $\mathbb{R}^n$ (see [9]).

A Representation for the Support Function. Let $\mathbf{B} \in \mathcal{B}^3_o$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of $\partial\mathbf{B}$. For $\omega \in \mathbf{S}^2$ we denote by $k_1(\omega), k_2(\omega)$ the principal normal curvatures of $\partial\mathbf{B}$ at $s(\omega)$. Let $k(\omega, \varphi)$ be the normal curvature in direction φ at $s(\omega)$ of $\partial\mathbf{B}$, φ is measured from the first principal direction. Denote by e_ω the plane containing the origin, which is orthogonal to ω.

Theorem 3. The support function of an origin symmetric 2-smooth convex body $\mathrm{B} \in \mathcal{B}^3_o$ has the following representation:

$$H(\xi) = (4\pi^2)^{-1} \int_{\mathrm{S}^2} \int_0^{2\pi} \sin^2 \alpha(\xi, \omega, \varphi) \, \frac{\sqrt{k_1(\omega) k_2(\omega)}}{k^2(\omega, \varphi)} d\varphi \, d\omega, \tag{9}$$

where $\alpha(\xi, \omega, \varphi)$ is the angle between $\varphi \in \mathbf{S}_\omega$ and the trace $e_\xi \cap e_\omega$.

Proof. We will need the following result from [10] (see also [4]): for any 2-smooth origin symmetric convex body $\mathbf{B}$, $\omega \in \mathbf{S}^2$ and $\varphi \in \mathbf{S}_\omega$, we have

$$\int_0^{2\pi} \sin^2 \alpha(\xi, \omega, \varphi) \, \frac{\sqrt{k_1(\omega) k_2(\omega)}}{k^2(\omega, \varphi)} \, d\varphi = \pi R(\omega, \varphi), \tag{10}$$

where φ is the direction of the projection ξ onto the tangent plane of $\partial\mathrm{B}$ at $s(\omega)$. Note that

$$R(\omega, \xi) = R(\omega, \varphi), \tag{11}$$

where $\varphi \in \mathbf{S}_\omega$.

For $n = 3$ we have (see (5))

$$4\pi H(\xi) = \int_{\mathbf{S}^2} R(\omega, \xi)\, \lambda_2(d\omega). \tag{12}$$

Substituting (10) into (12) and taking into account (11), we obtain the representation (9). □

Note that the representation (9) first was found in [11] (see also [4]) by means of a stochastic approximation of **B**.

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