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M a t h e m a t i c s

ON A FAMILY OF POLYNOMIALS WITH RESPECT TO THE HAAR SYSTEM

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We construct a sequence of polynomials with respect to the Haar system and show that they form democratic bases in $L^1(0,1)$.

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Introduction. Let $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ be a basis in a Banach space X. For a finite subset of natural numbers A, $A \subset \mathbb{N}$, denote

$$f_A = \sum_{i \in A} \psi_i.$$

In the case when X is a Hilbert space and Ψ is an orthonormal system one has $\|f_A\| = \sqrt{|A|}$, where $|A|$ denotes the cardinality of A. However, in the general case $\|f_A\|$ depends on $|A|$ as well as on the elements of A. In [1] it was introduced the term of democratic systems, namely:

D e f i n i t i o n . A set of elements $\Psi = \{\psi_n\}$ is called democratic system in X, if there exists a number $C \geq 1$, such that for any finity sets of integers A and B with $|A| = |B|$ the following relation holds

$$\|f_A\| \leq C\|f_B\|.$$

Democratic systems are related to the Greedy Algorithms. For details we refer to [2]. It is known that the Haar system is a democratic basis in $L^p(0,1)$ for $1 < p < \infty$. However, in $L^1(0,1)$ the Haar system is not democratic system. In [3] it was characterized all subsets of the Haar system that are democratic system in $L^1(0,1)$.

In this paper we construct a basis in $L^1(0,1)$ which is also democratic. Each element of that system is a polynomial with respect to the Haar system.

Let us remind the definition of the Haar system normalized in $L^1(0,1)$. Values of Haar functions at the points of discontinuity are not important for us, so we will

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change those values for the simplicity of definition. We denote $\Delta_1 = [0,1]$. A dyadic interval $\left[\frac{j-1}{2^i}, \frac{j}{2^i}\right)$; $j = 1,2,...,2^i,\ i = 0,1,2,...$, will be denoted by $\Delta_i^{(j)} = \Delta_{2^i+j}$. The left half of $\left[\frac{j-1}{2^i}, \frac{j}{2^i}\right)$ is the dyadic interval $\left[\frac{2j-2}{2^{i+1}}, \frac{2j-1}{2^{i+1}}\right)$ and the right half is $\left[\frac{2j-1}{2^{i+1}}, \frac{2j}{2^{i+1}}\right)$. Therefore, for any $n \geq 2$ we have

$$\Delta_n = \Delta_{2n-1} \cup \Delta_{2n}.$$

The Haar function associated with Δ_1 is the function $h_1 = h_{\Delta_1} \equiv 1$ and the Haar function associated with $\Delta_n,\ n = 2^i + j,\ j = 1,2,...,2^i,\ i = 0,1,2,...$, is

$$h_n(x) = h_i^{(j)}(x) = \begin{cases} 2^i & \text{for } x \in \Delta_{2n-1}; \\ -2^i & \text{for } x \in \Delta_{2n}; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$\int_0^1 h_n(t)dt = 0 \quad \text{and} \quad \int_0^1 |\, h_n(t) \,|\, dt = 1.$$

The Haar system is the set of functions $\mathbb{H} = \left\{h_n\right\}_{n=1}^{+\infty}$. The coefficients of expansion $c_n(f,\mathbb{H})$ are determined by the formulae

$$c_{[0,1]}(f,\mathbb{H}) = c_1(f,\mathbb{H}) = \int_0^1 f(x)dx,$$

$$c_{\Delta_n}(f,\mathbb{H}) = c_n(f,\mathbb{H}) = \int_{\Delta_{2n-1}} f(x)dx - \int_{\Delta_{2n}} f(x)dx, \quad n \geq 2. \tag{1}$$

Denote $\psi_i = h_i^{(2)}$ and let $\Phi = \left\{\phi_n : \quad n = 1,2,...\right\}$ be a system of functions consisting of the Haar functions $\mathbb{H} \setminus \left\{h_i^{(2)}\right\}_{i=1}^{+\infty}$ with the same order as in $\mathbb{H}$ (if $\phi_{m_1} = h_{n_1}$, $\phi_{m_2} = h_{n_2}$ and $m_1 < m_2$, then $n_1 < n_2$).

Let $\{M_i\}$ be a sequence of natural numbers. Denote $N_0 = 0$ and inductively define a sequence $N_i = N_{i-1} + M_i$. For a natural number i denote

$$f_{(i,0)} = \phi_i - \frac{1}{M_i + 1} \sum_{k=N_{i-1}+1}^{N_i} \psi_k,$$

and if $1 \leq j \leq M_i$, then

$$f_{(i,j)} = f_{(i,0)} + \psi_{N_{i-1}+j}.$$

In this paper we prove the following theorems.

Theorem 1. For any sequence of natural numbers $\{M_i\}$ the set of functions $\left\{\left\{f_{(i,j)}\right\}_{j=0}^{M_i}\right\}_{i=1}^{\infty}$ is a basis in $L^1(0,1)$.

Theorem 2. Let $\{M_i\}$ be a sequence of increasing natural numbers with $M_{i+1} \geq 2M_i$. Then the set of functions $\left\{\left\{f_{(i,j)}\right\}_{j=0}^{M_i}\right\}_{i=1}^{\infty}$ is a democratic basis in $L^1(0,1)$.

For the proof we adopt ideas in [4] and [5].

Proofs of the Theorems. Let $e_i = \{\delta_{ij}\}$ and let M_i, N_i, ϕ_i and ψ_i are defined as above. Denote $\sigma_n = [N_{n-1}+1, N_n] \bigcap \mathbb{N}$. One has $|\sigma_n| = M_n$. For a sequence $a \in l_1$ denote

$$m_n(a) = \sum_{i\in\sigma_n} a_i$$

and define operators $\mathcal{P}_n : l_1 \to l_1$ by

$$\mathcal{P}_n(a) = \frac{m_n(a)}{M_n} \sum_{i\in\sigma_n} e_i.$$

Finally, define

$$\mathcal{P}a = \sum_{n=1}^{\infty} \mathcal{P}_n a, \quad \mathcal{Q}a = a - \mathcal{P}a.$$

Its obvious that $\|\mathcal{P}\| = 1$. For $a \in c_{00}$ define

$$\|a\|_Y = \|Qa\|_{l_1} + \|\sum_n m_n(a)\phi_n\|_{L^1},$$

where c_{00} is the set of sequences with finite number of non zero terms. Completing this norm, we obtain a sequence space Y.

Theorem 3. The basic sequence $\{e_i\}$ is a Schauder basis for Y. Additionally, if $M_{i+1} \geq 2M_i$ for all $i \in \mathbb{N}$, then for any finite subset $A \subset \mathbb{N}$ one has

$$\frac{2\,|A|}{3} \leq \|\sum_{i\in A} e_i\|_Y \leq 3\,|A|\,. \tag{2}$$

Proof. Subspaces $\{e_i : i \in \sigma_n\}$ form a Schauder decomposition of Y. Since $\|\mathcal{P}\| = 1$, we have $\|\mathcal{Q}\| \leq 2$ and so each sequence $\{e_i : i \in \sigma_n\}$ in Y is 3 isomorphic to $\{e_i : i \in \sigma_n\}$ as a subspace in l_1. Thus $\{e_i\}$ is a basis in Y.

Now let us prove inequality (2).

The upper estimate follows from the bound

$$\begin{aligned}\|\sum_{i\in A} e_i\|_Y \leq & \|\mathcal{P}(\sum_{i\in A} e_i)\|_{l_1} + \|\sum_{i\in A} e_i\|_{l_1} + \|\sum_n m_n(\sum_{i\in A} e_i)\phi_n\|_{L^1} \leq \\ \leq & (1+\|P\|)\,|A| + \sum_n |\, m_n(\sum_{i\in A} e_i)\,| \leq 3\,|A|\,.\end{aligned}$$

To proceed the lower estimate, let us choose the biggest k such that $|A \cap \sigma_k| \geq \frac{M_k}{2}$. If such k does not exist, then we put $k = 0$. According to the monotonicity property of the Haar system, we have that

$$\|\sum_{i\in A} e_i\|_Y \geq \|\sum_n m_n(\sum_{i\in A} e_i)\phi_n\|_{L^1} \geq |\, m_k(\sum_{i\in A} e_i)\,| \geq \frac{M_k}{2}. \tag{3}$$

To complete the Proof of Theorem, we will need the following simple lemma immediately obtained from the definition of Q.

L e m m a . Let $B \subset \sigma_n$ and $|B| < \frac{|\sigma_n|}{2}$ for some natural n. Then

$$\left\| Q\left(\sum_{i\in B} e_i\right) \right\|_{l_1} \geq \frac{|B|}{2}.$$

Denote $D = \{i \in A \ : \ i \geq N_k\}$. According to the definition of k and Lemma, we have

$$\begin{aligned} \left\| Q\left(\sum_{i\in D} e_i\right) \right\|_{l_1} &= \sum_{n>k} \left\| Q\left(\sum_{i\in D\cap\sigma_n} e_i\right) \right\|_{l_1} \geq \\ &\geq \sum_{n>k} \frac{|D\cap\sigma_n|}{2} = \frac{|D|}{2} \geq \\ &\geq \frac{|A| - 2|\sigma_k|}{2} = \frac{|A| - 2M_k}{2}. \end{aligned} \tag{4}$$

In the last estimate we use the relation $N_k = M_1 + \ldots + M_k \leq 2M_k$. □

Combining (3) and (4), we get

$$\begin{aligned} \left\| \sum_{i\in A} e_i \right\|_Y &= \left\| Q(\sum_{i\in A} e_i) \right\|_{l_1} + \left\| \sum_n m_n(\sum_{i\in A} e_i)\phi_n \right\|_{L^1} \geq \\ &\geq \frac{M_k}{2} + \max\left(0, \frac{|A| - 2M_k}{2}\right) \geq \frac{2|A|}{3}. \end{aligned}$$ □

It is shown in [4], that there is an isomorphic operator $R: Y \to L^1(0,1)$ mapping the basic sequence e_i to the functions $f_{(i,j)}$. So, Theorems 1 and 2 are proved.

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