PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 2, p. 39–47

Informatics

ON λ-DEFINABILITY OF ARITHMETICAL FUNCTIONS WITH INDETERMINATE VALUES OF ARGUMENTS

S. A. NIGIYAN [*]

Chair of Programming and Information Technologies YSU, Armenia

In this paper the arithmetical functions with indeterminate values of arguments are regarded. It is known that every λ-definable arithmetical function with indeterminate values of arguments is monotonic and computable. The λ-definability of every computable, monotonic, 1-ary arithmetical function with indeterminate values of arguments is proved. For computable, monotonic, k-ary, $k \geq 2$, arithmetical functions with indeterminate values of arguments, the so-called diagonal property is defined. It is proved that every computable, monotonic, k-ary, $k \geq 2$, arithmetical function with indeterminate values of arguments, which has the diagonal property, is not λ-definable. It is proved that for any $k \geq 2$, the problem of λ-definability for computable, monotonic, k-ary arithmetical functions with indeterminate values of arguments is algorithmic unsolvable. It is also proved that the problem of diagonal property of such functions is algorithmic unsolvable, too.

MSC2010: 68Q01, 68Q05.

***Keywords*:** arithmetical functions, indeterminate value of argument, monotonicity, computability, strong computability, λ-definability, algorithmic unsolvability.

Introduction. The paper is devoted to arithmetical functions with indeterminate values of arguments. These functions are defined on partially ordered set $M = N \cup \{\perp\}$, where N is the set of natural numbers, $\perp$ is the element, which corresponds to indeterminate value. Each element of M is comparable with itself and with $\perp$, which is the least element of M. The notion of monotonic function is introduced in a conventional way. A function is said to be naturally extended, if its value is $\perp$ whenever the value at least one of the argument is $\perp$. Such functions were regarded in [1]. In [2] the notions of computability, strong computability, λ-definability for arithmetical functions with indeterminate values of arguments were introduced. It was proved, that every λ-definable arithmetical function with indeterminate values of arguments is monotonic and computable. It was proved that

---

[*] E-mail: nigiyan@ysu.am

every computable, naturally extended arithmetical function with indeterminate values of arguments is λ-definable. It was proved too, that there exist strong computable, monotonic, not naturally extended arithmetical functions with indeterminate values of arguments, which are not λ-definable.

In this paper it is proved that every computable, monotonic, 1-ary arithmetical function with indeterminate values of arguments is λ-definable. For computable, monotonic, k-ary, $k \geq 2$, arithmetical functions with indeterminate values of arguments, the so-called diagonal property is defined. It is proved that every computable, monotonic, k-ary, $k \geq 2$, arithmetical function with indeterminate values of arguments, which has the diagonal property, is not λ-definable. The examples of λ-definable and not λ-definable strong computable, monotonic, not naturally extended, arithmetical functions with indeterminate values of arguments are given. It is proved that for any $k \geq 2$, the problem of λ-definability for strong computable (therefore, for computable), monotonic, k-ary arithmetical functions with indeterminate values of arguments is algorithmic unsolvable. It is also proved that the problem of diagonal property for such functions is algorithmic unsolvable, too. It is proved that for any $k \geq 1$, the problem of monotonicity for strong computable (therefore, for computable), k-ary arithmetical functions with indeterminate values of argument is algorithmic unsolvable.

Definitions Used and Previous Results. In this section definitions and previous results are given, which as a rule, are borrowed from [2, 3]. These definitions and results will be accompanied by some comments.

Let $M = N \cup \{\perp\}$, where $N = \{0, 1, 2, \ldots\}$ is the set of natural numbers, $\perp$ is the element which corresponds to indeterminate value. Let us introduce the partial ordering $\subseteq$ on the set M. For every $m \in M$ we have: $\perp \subseteq m$ and $m \subseteq m$. A mapping $\varphi : M^k \to M$, $k \geq 1$, is said to be k-ary arithmetical function with indeterminate values of arguments.

Definition 1. A function $\varphi : M^k \to M$, $k \geq 1$, is said to be computable if there exists an algorithm (Turing machine, see [4]), which for all $m_1, \ldots, m_k \in M$ stops with value $\varphi(m_1, \ldots, m_k)$ if $\varphi(m_1, \ldots, m_k) \neq \perp$, and stops with value $\perp$, or works infinitely if $\varphi(m_1, \ldots, m_k) = \perp$.

Definition 2. A function $\varphi : M^k \to M$, $k \geq 1$, is said to be strong computable, if there exists an algorithm (Turing machine, see [4]), which stops with value $\varphi(m_1, \ldots, m_k)$ for all $(m_1, \ldots, m_k) \in M$.

It is obvious, that every strong computable, arithmetical function with indeterminate values of arguments is computable, but not every computable arithmetical function with indeterminate values of arguments is strong computable.

Definition 3. A function $\varphi : M^k \to M$, $k \geq 1$, is said to be monotonic if $(m_1, \ldots, m_k) \subseteq (\mu_1, \ldots, \mu_k)$ implies $\varphi(m_1, \ldots, m_k) \subseteq \varphi(\mu_1, \ldots, \mu_k)$ for all $m_i, \mu_i \in M$, $i \in 1, \ldots, k$.

Let $\varphi : M^k \to M$, $k \geq 1$, be arithmetical function with indeterminate values of arguments. One can see that φ is monotonic $\Leftrightarrow$ if for all $m_1, ..., m_i, ..., m_k \in M$, we have: if for some $i = 1, \ldots, k$, $m_i = \perp$ and $\varphi(m_1, \ldots, \perp, \ldots, m_k) \neq \perp$, then

$\varphi(m_1,\dots,\perp,\dots,m_k)=\varphi(m_1,\dots,n,\dots,m_k)$ for all $n\in N$.

Definition 4. A function $\varphi: M^k\to M,\ k\geq 1$, is said to be naturally extended if for all $m_1,\dots,m_k\in M$, we have: if for some $i=1,\dots,k,\ m_i=\perp$, then $\varphi(m_1,\dots,m_k)=\perp$.

It is easy to see that every naturally extended arithmetical function with indeterminate values of arguments is monotonic.

Let us fix countable set of variables V and define the set of terms Λ :

1. if $x\in V$, then $x\in\Lambda$;
2. if $t_1,t_2\in\Lambda$, then $(t_1t_2)\in\Lambda$;
3. if $x\in V$ and $t\in\Lambda$, then $(\lambda xt)\in\Lambda$.

Abridged notation for the terms will be used: term $(\dots(t_1t_2)\dots t_k)$, where $t_i\in\Lambda$, $i=1,\dots,k$, $k>1$, is denoted as $t_1t_2\dots t_k$, and term $(\lambda x_1(\lambda x_2(\dots(\lambda x_nt)\dots)))$, where $x_j\in V$, $t\in\Lambda$, $j=1,\dots,n$, $n>0$, is denoted as $\lambda x_1x_2\dots x_n.t$.

The notion of a free and bound occurrence of a variable in a term and the notion of a free variable of a term are introduced in a conventional way. A term that does not contain free variables is said to be closed.

Terms t_1 and t_2 are said to be congruent (is denoted as $t_1\equiv t_2$) if one term can be obtained from the other by renaming bound variables. In what follows, congruent terms are considered identical.

The term obtained from a term t as a result of the simultaneous substitution of a term τ instead of all free occurrences of a variable x is denoted as $t[x:=\tau]$. A substitution is said to be admissible if all free occurrences of variables of the term being substituted remain free after the substitution. We will consider only admissible substitutions.

Let us remind the notion of the β-reduction:

$$\beta=\{((\lambda x.t)\tau,t[x:=\tau])\ \big|\ t,\tau\in\Lambda,\ x\in V\}.$$

A one-step β-reduction ($\to_\beta$), β-reduction ($\to\!\!\!\to_\beta$) and β-equality ($=_\beta$) are defined in a standard way.

We remind that the term $(\lambda x.t)\tau$ is referred to as β-redex. A term not containing β-redexes is referred to as β-normal form (further, simply normal form). The set of all normal forms is denoted as NF. A term t is said to have a normal form, if there exists a term $t'\in$NF such that $t=_\beta t'$. A term of the form $\lambda x_1x_2\dots x_n.xt_1t_2\dots t_k$, where $x,x_i\in V$, $t_j\in\Lambda$, $i=1,\dots,n$, $n\geq 0$, $j=1,\dots,k$, $k\geq 0$, is referred to us a head normal form. The set of all head normal forms is denoted by HNF. A term t is said to have a head normal form, if there exists a term $t'\in$ HNF such that $t=_\beta t'$. It is known that NF $\subset$ HNF, but HNF $\not\subset$ NF.

We will extensively use the corollary from the Church–Rosser theorem, which says that for any term $t\in\Lambda$ the following two assertions are valid:

1. $t=_\beta t',\ t'\in\text{NF}\Rightarrow t\to\!\!\!\to_\beta t'$,
2. $t=_\beta t',\ t=_\beta t'',\ t',\ t''\in\text{NF}\Rightarrow t'\equiv t''$.

Remind the following statement: if $t=_\beta t'$ and $t'\in$ NF, then $t\to\!\!\!\to_\beta t'$ and $\to\!\!\!\to_\beta$ is the left β-reduction (i.e. the β-reduction where, each time, the leftmost

β-redex is taken). We will also use the following statement: If term $t \in \Lambda$ has not head normal form, then a term $t\tau$ has not head normal form too, where $\tau \in \Lambda$.

We introduce the following notations for some terms to be used in below: $I \equiv \lambda x.x$, $T \equiv \lambda xy.x$, $F \equiv \lambda xy.y$, $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, $\underline{\text{if}}\ t_1\ \underline{\text{then}}\ t_2\ \underline{\text{else}}\ t_3 \equiv t_1t_2t_3$, Zero$\equiv \lambda x.xT$, $\langle \perp \rangle \equiv \Omega, \langle 0 \rangle \equiv I, \langle n+1 \rangle \equiv \lambda x.xF\langle n \rangle$, where $x,y \in V$, $t_1,t_2,t_3 \in \Lambda$, $n \in N$. It is easy to see that: the term Ω does not have a head normal form, $\underline{\text{if}}\ T\ \underline{\text{then}}\ t_2\ \underline{\text{else}}\ t_3 =_\beta t_2$, $\underline{\text{if}}\ F\ \underline{\text{then}}\ t_2\ \underline{\text{else}}\ t_3 =_\beta t_3$, Zero$\langle 0 \rangle =_\beta T$, Zero$\langle n+1 \rangle =_\beta F$, Zero$\langle \perp \rangle$ does not have a head normal form, term $\langle n \rangle$ is closed normal form, and if $n_1 \neq n_2$, then $\langle n_1 \rangle$ and $\langle n_2 \rangle$ are not congruent terms, where $n, n_1, n_2 \in N$.

Definition 5. A function $\varphi : M^k \to M$, $k \geq 1$, is said to be λ-definable if there exists such term $\Phi \in \Lambda$, that for all $m_1, \dots, m_k \in M$ we have:

$\Phi\langle m_1 \rangle \dots \langle m_k \rangle =_\beta \langle \varphi(m_1, \dots, m_k) \rangle$, if $\varphi(m_1, \dots, m_k) \neq \perp$ and

$\Phi\langle m_1 \rangle \dots \langle m_k \rangle$ does not have a head normal form, if $\varphi(m_1, \dots, m_k) = \perp$.

The term Φ is said to be the term which λ-defines the function φ.

On λ-Definability of Computable, Monotonic, Arithmetical Functions with Indeterminate Values of Arguments. In [2] it was shown, that every λ-definable arithmetical function with indeterminate values of arguments is monotonic and computable. Therefore, exploring the λ-definability of arithmetical functions with indeterminate values of arguments, we consider the set of computable, monotonic arithmetical functions with indeterminate values of arguments.

Theorem 1. Every computable, monotonic, 1-ary arithmetical function with indeterminate values of arguments is λ-definable.

Proof. Let $\varphi : M \to M$ be computable, monotonic 1-ary arithmetical function with indeterminate values of arguments. If $\varphi(\perp) = \perp$, then φ will be computable, naturally extended arithmetical function with indeterminate values of arguments, and from the [2] follows, that φ will be λ-definable. If $\varphi(\perp) = n$, where $n \in N$, then $\varphi(m) = n$, for all $m \in M$, because the function φ is monotonic, and term $\Phi \equiv \lambda x.\langle n \rangle$, where $x \in V$, λ-defines the function φ. $\square$

Theorem 2. Every computable, monotonic, k-ary $(k \geq 2)$ arithmetical function with indeterminate values of arguments $\varphi : M^k \to M$ is not λ-definable if φ satisfies the following conditions: $\varphi(\perp, \perp, \dots, \perp) = \perp$ and there exist such natural number s, $2 \leq s \leq k$, and such sequences of values of arguments of function φ

$$\begin{gathered} m_{11}, m_{12}, \dots, m_{1s}, m_{s+1}, \dots, m_k \\ m_{21}, m_{22}, \dots, m_{2s}, m_{s+1}, \dots, m_k \\ \dots \\ m_{s1}, m_{s2}, \dots, m_{ss}, m_{s+1}, \dots, m_k, \end{gathered}$$

where $m_{ij} \in M$, $m_{ii} = \perp$, $i,j = 1, \dots, s$, $m_{s+r} \in M$, $r = 1, \dots, k-s$, that $\varphi(\perp, \perp, \dots, \perp, m_{s+1}, \dots, m_k) = \perp$ and

$$\begin{gathered} \varphi(\perp, m_{12}, \dots, m_{1s}, m_{s+1}, \dots, m_k) \neq \perp \\ \varphi(m_{21}, \perp, \dots, m_{2s}, m_{s+1}, \dots, m_k) \neq \perp \\ \dots \\ \varphi(m_{s1}, m_{s2}, \dots, \perp, m_{s+1}, \dots, m_k) \neq \perp. \end{gathered}$$

This property of computable, monotonic, k-ary ($k \geq 2$) arithmetical functions with indeterminate values of arguments, will be called the diagonal property of such functions.

Proof. Let the function φ has the diagonal property. Let us show that the function φ is not λ-definable. Assume that the function φ is λ-definable and a term Φ λ-defines the function φ. We define the function $\psi : M^s \to M$ as follows: for all $m_1, m_2, \dots, m_s \in M$, $\psi(m_1, m_2, \dots, m_s) = \varphi(m_1, m_2, \dots, m_s, m_{s+1}, \dots, m_k)$. It is easy to see that ψ is computable, monotonic, s-ary ($s \geq 2$) arithmetical function with indeterminate values of arguments for which $\psi(\perp, \perp, \dots, \perp) = \perp$ and

$$\begin{gathered}\psi(\perp, m_{12}, \dots, m_{1s}) \neq \perp\\ \psi(m_{21}, \perp, \dots, m_{2s}) \neq \perp\\ \dots\\ \psi(m_{s1}, m_{s2}, \dots, \perp) \neq \perp.\end{gathered}$$

Since the term Φ, by hypothesis, λ-defines the function φ, the term $\Psi \equiv \lambda x_1 \dots x_s.\Phi x_1 \dots x_s \langle m_{s+1} \rangle \dots \langle m_k \rangle$, where $x_i \in V, i \neq j \Rightarrow x_i \neq x_j, i, j = 1, \dots, s$, λ-defines the function ψ. Let us regard the term $\Psi y_1 \dots y_s$, where $y_1, \dots, y_s$ are pairwise distinct variables that are not used in the term Ψ. Since $\psi(\perp, m_{12}, \dots, m_{1s}) \neq \perp$, the term $\Psi\Omega\langle m_{12} \rangle \dots \langle m_{1s} \rangle$ has a closed normal form and by the left β-reduction of the term $\Psi y_1 \dots y_s$, we cannot get a term t, in which y_1 is the leftmost occurrence of a free variable, which is on the left of the leftmost β-redex of the term t (otherwise, the term $\Psi\Omega\langle m_{12} \rangle \dots \langle m_{1s} \rangle$ will not have a normal form). Further, since $\psi(m_{21}, \perp, \dots, m_{2s}) \neq \perp$, the term $\Psi\langle m_{21} \rangle \Omega \langle m_{23} \rangle \dots \langle m_{2s} \rangle$ has a closed normal form and by the left β-reduction of the term $\Psi y_1 \dots y_s$, we cannot get a term t, in which y_2 is the leftmost occurrence of a free variable, which is on the left of the leftmost β-redex of the term t (otherwise, the term $\Psi\langle m_{21} \rangle \Omega \langle m_{23} \rangle \dots \langle m_{2s} \rangle$ will not have a normal form) and so on. Finally, since $\psi(m_{s1}, m_{s2}, \dots, m_{ss-1}, \perp) \neq \perp$, the term $\Psi\langle m_{s1} \rangle \langle m_{s2} \rangle \dots \langle m_{ss-1} \rangle \Omega$ has a closed normal form and by the left β-reduction of the term $\Psi y_1 \dots y_s$ we cannot get a term t, in which y_s is the leftmost occurrence of a free variable, which is on the left of the leftmost β-redex of the term t (otherwise, the term $\Psi\langle m_{s1} \rangle \langle m_{s2} \rangle \dots \langle m_{ss-1} \rangle \Omega$ will not have a normal form). Thus, by the left β-reduction of the term $\Psi y_1 \dots y_s$ we can get a closed normal form. Therefore, by the left β-reduction of the term $\Psi\Omega\Omega\dots\Omega$ we can get the same closed normal form. Contradiction, since $\psi(\perp, \perp, \dots, \perp) = \perp$ and the term $\Psi\Omega\Omega\dots\Omega$ does not have a normal form. Therefore, the function φ is not λ-definable. $\square$

Consider the functions $cond : M^3 \to M$ and $g : M^3 \to M$, for all $m_1, m_2, m_3 \in M$ we have:

$$cond(m_1, m_2, m_3) = \begin{cases} m_2, & \text{if } m_1 \neq \perp,\ m_1 \geq 1 \text{ or } m_2 = m_3,\\ m_3, & \text{if } m_1 \neq \perp,\ m_1 = 0 \text{ or } m_2 = m_3,\\ \perp, & \text{otherwise;} \end{cases}$$

$$g(m_1, m_2, m_3) = \begin{cases} 0, & \text{if } m_1 = 0,\ m_3 \neq \perp,\ m_3 \geq 1 \text{ or}\\ & \quad m_2 \neq \perp,\ m_2 \geq 1,\ m_3 = 0 \text{ or}\\ & \quad m_1 \neq \perp,\ m_1 \geq 1,\ m_2 = 0,\\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that $cond$ and g are strong computable, monotonic arithmetical functions with indeterminate values of arguments. It is also easy to see that the functions $cond$ and g have diagonal property, since $cond(\perp,\perp,\perp) = \perp$, $cond(\perp,\perp,0) = \perp$, $cond(\perp,0,0) = 0$, $cond(0,\perp,0) = 0$, here $k = 3$, $s = 2$, and $g(\perp,\perp,\perp) = \perp$, $g(\perp,1,0) = 0$, $g(0,\perp,1) = 0$, $g(1,0,\perp) = 0$, here $k = s = 3$. Therefore, the functions $cond$ and g are not λ-definable.

Now we give examples of strong computable, monotonic, not naturally extended, arithmetical functions with indeterminate values of arguments $if : M^3 \to M$ and $h : M^3 \to M$, which have no diagonal property and are λ-definable. For all $m_1, m_2, m_3 \in M$ we have:

$$if(m_1,m_2,m_3) = \begin{cases} m_2, & \text{if } \ m_1 \neq \perp,\ m_1 \geq 1, \\ m_3, & \text{if } \ m_1 \neq \perp,\ m_1 = 0, \\ \perp, & \text{if } \ m_1 = \perp; \end{cases}$$

$$h(m_1,m_2,m_3) = \begin{cases} 0, & \text{if } \ m_1 = 0,\ m_3 \neq \perp,\ m_3 \geq 1 \text{ or} \\ & \quad m_2 \neq \perp,\ m_2 \geq 1,\ m_3 = 0, \\ \perp, & \text{otherwise.} \end{cases}$$

The following terms If and H, λ-define the functions if and h respectively:

$$\text{If} \ \equiv \lambda xyz.(\text{Zero}\, x)zy,$$

$$\text{H} \ \equiv \lambda xyz.\ \underline{\text{if}}\ \text{Zero}\ z\ \underline{\text{then}}(\underline{\text{if}}\ \text{Zero}\ y\ \underline{\text{then}}\ \Omega\ \underline{\text{else}}\langle 0\rangle)\underline{\text{else}}(\underline{\text{if}}\ \text{Zero}\ x\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\Omega).$$

We formulate a corollary of Theorem 2, which is a special case of Theorem 2 for $k = 2$.

Corollary 1 (Theorem 2). Every computable, monotonic, 2-ary arithmetical function with indeterminate values of arguments $\varphi : M^2 \to M$, for which $\varphi(\perp,\perp) = \perp$ and there exist such $n_1, n_2 \in N$, that $\varphi(\perp,n_2) \neq \perp$ and $\varphi(n_1,\perp) \neq \perp$, is not λ-definable.

Consider functions $mul : M^2 \to M$, $\wedge : M^2 \to M$ and $\vee : M^2 \to M$, for all $m_1, m_2 \in M$, we have:

$$mul(m_1,m_2) = \begin{cases} 0, & \text{if } \ m_1 = 0, \text{ or } \ m_2 = 0, \\ m_1 * m_2, & \text{if } m_1 \neq \perp, \text{ and } \ m_2 \neq \perp, \\ \perp, & \text{otherwise;} \end{cases}$$

$$\wedge(m_1,m_2) = \begin{cases} 0, & \text{if } m_1 = 0, \text{ or } \ m_2 = 0, \\ 1, & \text{if } m_1 \neq \perp,\ m_2 \neq \perp, \text{ and } \ m_1 \geq 1,\ m_2 \geq 1, \\ \perp, & \text{otherwise;} \end{cases}$$

$$\vee(m_1,m_2) = \begin{cases} 0, & \text{if } m_1 = 0, \text{ and } m_2 = 0, \\ 1, & \text{if } m_1 \neq \perp,\ m_1 \geq 1, \text{ or } m_2 \neq \perp, m_2 \geq 1, \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that mul, $\wedge$ and $\vee$ are strong computable, monotonic arithmetical functions with indeterminate values of arguments. It is also easy to see that the functions mul, $\wedge$ and $\vee$ have the diagonal property, since $mul(\perp,\perp) = \perp$,

$mul(\perp,0)=0$, $mul(0,\perp)=0$, $\wedge(\perp,\perp)=\perp$, $\wedge(\perp,0)=0$, $\wedge(0,\perp)=0$, and $\vee(\perp,\perp)=\perp$, $\vee(\perp,1)=1$, $\vee(1,\perp)=1$. Therefore, the functions $mul,\wedge$ and $\vee$ are not λ-definable.

Now we give examples of strong computable, monotonic, not naturally extended, arithmetical functions with indeterminate values of arguments $andl : M^2 \to M$ and $andr : M^2 \to M$, which have no diagonal property and are λ-definable. For all $m_1, m_2 \in M$ we have:

$$andl(m_1,m_2)=\begin{cases} 0, & \text{if } m_1=0, \text{ or } m_1\neq\perp,\ m_1\geq 1,\ m_2=0,\\ 1, & \text{if } m_1\neq\perp,\ m_2\neq\perp \text{ and } m_1\geq 1,\ m_2\geq 1,\\ \perp, & \text{otherwise;}\end{cases}$$

$$andr(m_1,m_2)=\begin{cases} 0, & \text{if } m_2=0, \text{ or } m_2\neq\perp,\ m_2\geq 1,\ m_1=0,\\ 1, & \text{if } m_1\neq\perp,\ m_2\neq\perp \text{ and } m_1\geq 1,\ m_2\geq 1,\\ \perp, & \text{otherwise.}\end{cases}$$

The following terms Andl and Andr, λ-define the functions *andl* and *andr* respectively:

$$\text{Andl} \equiv \lambda xy.\underline{\text{if}}\ \text{Zero}\ x\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\ (\underline{\text{if}}\ \text{Zero}\ y\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\langle 1\rangle),$$

$$\text{Andr} \equiv \lambda xy.\underline{\text{if}}\ \text{Zero}\ y\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\ (\underline{\text{if}}\ \text{Zero}\ x\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\langle 1\rangle).$$

Now we give examples of strong computable, monotonic, not naturally extended, arithmetical functions with indeterminate values of arguments $orl : M^2 \to M$ and $orr : M^2 \to M$, which have no diagonal property and are λ-definable. For all $m_1, m_2 \in M$ we have:

$$orl(m_1,m_2)=\begin{cases} 0, & \text{if } m_1=0 \text{ and } m_2=0,\\ 1, & \text{if } m_1\neq\perp,\ m_1\geq 1 \text{ or } m_1=0,\ m_2\neq\perp,\ m_2\geq 1,\\ \perp, & \text{otherwise;}\end{cases}$$

$$orr(m_1,m_2)=\begin{cases} 0, & \text{if } m_1=0 \text{ and } m_2=0,\\ 1, & \text{if } m_2\neq\perp,\ m_2\geq 1 \text{ or } m_2=0,\ m_1\neq\perp,\ m_1\geq 1,\\ \perp, & \text{otherwise.}\end{cases}$$

The following terms Orl and Orr, λ-define the functions *orl* and *orr* respectively:

$$\text{Orl} \equiv \lambda xy.\underline{\text{if}}\ \text{Zero}\ x\ \underline{\text{then}}\ (\underline{\text{if}}\ \text{Zero}\ y\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\langle 1\rangle)\underline{\text{else}}\langle 1\rangle,$$

$$\text{Orr} \equiv \lambda xy.\underline{\text{if}}\ \text{Zero}\ y\ \underline{\text{then}}\ (\underline{\text{if}}\ \text{Zero}\ x\ \underline{\text{then}}\langle 0\rangle\underline{\text{else}}\langle 1\rangle)\underline{\text{else}}\langle 1\rangle.$$

Algorithmic Problems. Speaking about the algorithmic problems for computable arithmetical functions with indeterminate values of arguments, we believe that each of them is given by its algorithm.

Theorem 3. The λ-definability problem for strong computable, monotonic, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 2$.

Proof. Let $T_0, T_1, \dots T_n, \dots$ be an effective numeration of Turing machines (see [4]), $n \in N$. For each $n \in N$ we define the function $f_n : M^k \to M,\ k \geq 2$, by describing its algorithm. For all $m_1, m_2, \dots, m_k \in M$ we have:

$$f_n(m_1, m_2 \dots, m_k) = \begin{cases} 0, & \text{if } m_1 \neq \perp, m_2 = \perp \text{ and Turing machine } T_n \\ & \text{halts on 0 after } \leq m_1 \text{ steps, or} \\ & \text{if } m_1 = \perp, m_2 \neq \perp \text{ and Turing machine } T_n \\ & \text{halts on 0 after } \leq m_2 \text{ steps, or} \\ & \text{if } m_1 \neq \perp, m_2 \neq \perp \text{ and Turing machine } T_n \\ & \text{halts on 0 after } \leq \max(m_1, m_2) \text{ steps,} \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that for any $n \in N$ f_n is strong computable, monotonic arithmetical function with indeterminate values of arguments. If Turing machine T_n halts on 0, than the function f_n has the diagonal property, since there exists such $n_1 \in N$, that for all $m_3, \dots, m_k \in M, f_n(\perp, \perp, m_3, \dots, m_k) = \perp$, $f_n(\perp, n_1, m_3, \dots, m_k) = 0$, $f_n(n_1, \perp, m_3, \dots, m_k) = 0$ and, therefore, f_n is not λ-definable. If Turing machine T_n does not halt on 0, than for all $m_1, \dots, m_k \in M$, $f_n(m_1, \dots, m_k) = \perp$, and the term $\Phi \equiv \lambda x_1 \dots x_k.\Omega$ λ-defines the function f_n, therefore, f_n is λ-definable. Thus, the assumption of the solvability of the λ-definability problem for strong computable, monotonic, k-ary ($k \geq 2$) arithmetical functions with indeterminate values of arguments, would lead to the solvability of the halting problem of Turing machines. □

Corollary 2 (Theorem 3). The λ-definability problem for computable, monotonic, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 2$.

Theorem 4. The diagonal property for strong computable, monotonic, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 2$.

Proof. The proof repeats the proof of Theorem 3. It is easy to see, that for any $n \in N$ we have: Turing machine T_n halts on 0 $\Leftrightarrow$ function f_n has the diagonal property. Thus, the assumption of the solvability of the diagonal property for strong computable, monotonic, k-ary ($k \geq 2$) arithmetical functions with indeterminate values of arguments, would lead to the solvability of the halting problem of Turing machines. □

Corollary 3 (Theorem 4). The diagonal property for computable, monotonic, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 2$.

Theorem 5. The monotonicity property for strong computable, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 1$.

Proof. Let $T_0, T_1, \dots, T_n, \dots$ be an effective numeration of Turing machines (see [4]), $n \in N$. For each $n \in N$ we define the function $f_n : M^k \to M,\ k \geq 1$, by describing its algorithm. For all $m_1, m_2, \dots, m_k \in M$ we have:

$$f_n(m_1, \dots, m_k) = \begin{cases} 0, & \text{if } m_1 = \perp \text{ or } m_1 \neq \perp \text{ and Turing machine } T_n \\ & \text{does not halt on 0 after } m_1 \text{ steps,} \\ \perp, & \text{otherwise.} \end{cases}$$

It is easy to see that for any $n \in N$ f_n is strong computable, arithmetical function with indeterminate values of arguments. If Turing machine T_n does not halt on 0, than for all $m_1,\ldots,m_k \in M$, $f_n(m_1,\ldots,m_k) = 0$ and, obviously, the function f_n is monotonic. If Turing machine T_n halts on 0, than there exists such $n_1 \in N$, that for all $m_2,\ldots,m_k \in M$, $f_n(\perp,m_2,\ldots,m_k) = 0$, $f_n(n_1,m_2,\ldots,m_k) = \perp$ and, obviously, the function f_n is not monotonic. Thus, the assumption of the solvability of the monotonicity property for strong computable, monotonic, k-ary ($k \geq 1$) arithmetical functions with indeterminate values of arguments, would lead to the solvability of the halting problem of Turing machines. ☐

Corollary 4 (Theorem 5). The monotonicity property for computable, k-ary arithmetical functions with indeterminate values of arguments is unsolvable for any $k \geq 1$.

Received 22.01.2016

REFERENCES

1. **Manna Z.** Mathematical Theory of Computation. McGraw-Hill Book Company, 1974.
2. **Nigiyan S.A.** On Non-classical Theory of Computability. // Proceedings of the YSU. Physical and Mathematical Sciences, 2015, № 1, p. 52–60.
3. **Barendregt H.** The Lambda Calculus. Its Syntax and Semantics. North-Holland Publishing Company, 1981.
4. **Rogers H.** Theory of Recursive Functions and Effective Computability. McGraw-Hill Book Company, 1967.