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Mathematics

CONSTANT WEIGHT PERFECT AND *D*-REPRESENTABLE CODES

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The problem of the existence of non trivial constant weight perfect codes in the B^n -space defined over $GF(2)$ remains unsolved up to now. It has been proved in the present paper that the problem of the existence of constant weight perfect codes is equivalent to the problem of the existence of *D*-representable codes in the fixed layer.

***Keywords*:** constant weight perfect codes, space splitting, Dirichlet regions, *D*-representable codes.

1. Introduction. Many problems of the coding theory, combinatorial analysis, image recognition as well as expert assessments can be viewed as splitting problems for the *n*-dimensional space over $GF(2)$. For instance, the space splitting with the help of perfect sets, i.e. space splitting into spheres of similar radii is well-known. However, it is turns out that the class of perfect splitting is not that "rich" (see [1–2]). So we have to deal with the splitting of space not necessarily into spheres. The examples of such sets, besides the perfect ones, are *k*-dimensional subcubes. In that case Dirichlet regions are "projection" sets.

The full description of the class of sets that splitting *n*-dimensional space over $GF(2)$ is obtained in the paper [3], and this result can be considered as a final one.

In the present paper we are going to consider the problem of splitting of the sets of constant weights (with the elements of fixed weight) of the n-dimensional space over the $GF(2)$ in the Dirichlet region.

The interest on this problem is motivated by its applications: the investigation of many problems in *n*-dimensional space (the estimates of the code power, the problems of packing and covering, the t-schemes, etc.) use the results, received in the subsets with constant weights of that space.

The Statement of the Problem and the Main Result. Let us define $B^n = \{0,1\}^n$ to be the n-dimensional vector space over $GF(2)$. We assume that consider the distance of the space B^n is the Hemming distance $\rho(x,y)$, which is defined in the following way: $\rho(x,y)=\sum_{i=1}^{n}(\alpha_i \oplus \beta_i)$, where $\oplus$ is the addition in

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modulo 2 , $x=(\alpha_1\alpha_2...\alpha_n)$, $y=(\beta_1\beta_2...\beta_n)$.

Geometrically the metric space B^n corresponds to the set of vertices of the n-dimensional unit cube, and the distance between the points from B^n is equal to the minimal number of edges in the chain, connecting the corresponding vertices of the cube.

Let A be any subset of B^n and φ by an arbitrary function defined on the set of natural numbers. For $x\in A$ we define the Dirichlet region $D_x(\varphi)$ in the following way: $D_x(\varphi)=\{y\mid y\in B^n,\varphi(\rho(x,y))\le\varphi(\rho(z,y)),\forall z\in A\}$. It's clear that $\bigcup\limits_{x\in A}D_x(\varphi)=B^n$. For $\varphi(k)=k$ we have

$$D_x(\varphi)=D_x=\{y\mid y\in B^n,\rho(x,y)\le\rho(z,y),\forall z\in A\}.$$

We mention the following simple properties of $D_x(\varphi)$. Let φ_1 be a monotonically increasing function, and let φ_2 be a monotonically decreasing function. In that case:

1. $D_x(\varphi_1)=D_x$ and Dirichlet region of the point x contains all the points of B^n, situated not farther from x than from other points of A.
2. $D_x(\varphi_2)=D_x(n-\rho)$ and Dirichlet region of the point x contains all the points of B^n, situated not closer from x than from other points of A.

Suppose $B_k^n=\{y\mid y\in B^n,\rho(0^n,y)=k\}$.

Definition 1. A set $A\subseteq B_k^n$ is called $D(\varphi)$-representable code (D-representable code), if Dirichlet regions $D_x(\varphi)$ (D_x) of all a points of A are pairwise disjoint in B_k^n. Taking into account the fact that for a monotone function φ we have $y\in D_{x_1}(\varphi)\bigcap D_{x_2}(\varphi)$, $x_1,x_2\in A$, $x_1\ne x_2$, iff $\rho(x_1,y)=\rho(x_2,y)$, we come to

Lemma 1. If φ is a strictly monotone function, then the set A is $D(\varphi)$-representable, iff A is a D-representable.

This means that the splittings for different strictly monotone function φ are equivalent. In the light of accordance with this Lemma, our task is to find a necessary and sufficient condition for of D-representable codes for existence.

Lemma 2. If for $x,y,z\in B_k^n$ we have equality $\rho(x,y)=\rho(x,z)$, then there exists a sequence of points $y=y_1,y_2,...,y_l=z$ in B_k^n satisfying

$$\rho(y_i,y_{i+1})=2,\ \rho(x,y_{i+1})=\rho(x,y),\ i=1,2,...,l-1.$$

Let $S_t(x)$ be the sphere with centre x and radius t, i.e. $S_t(x)$ is the set of all $y\in B^n$ satisfying $\rho(x,y)\le t$.

Definition 2 [4]. A set $A\subseteq B_k^n$ is called constant weight perfect code (constant weight perfect code of radius t), if

1. $B_k^n\subseteq\bigcup\limits_{x\in A}S_t(x)$;
2. $S_t(x)\bigcap S_t(y)\bigcap B_k^n=\emptyset$ for all x,y from A.

We define $Q(A)$ to by the subcube of the minimal dimension containing A, and for $x \in A$ the set $M(x)$ is defined in the following way:

$$M(x) = \{y \mid y \in A; Q(\{x, y\}) \subseteq D_x \bigcup D_y\}.$$

Lemma 3. For any points $x, x_1 \in A$ there exists a sequence of points $x_1, x_2, ..., x_l = x$ in A such that $x_i \in M(x_{i-1})$, $i = 2,3,...,l$.

Proof. Suppose $C \subset A$ is the maximal (or one of the maximals) set, sharing the condition of the Lemma. Assume $y_0 \in C$, $x_0 \in A \setminus C$ and

$$\rho(x_0, y_0) = \min \rho(z_1, z_2), \ z_1 \in C, z_2 \in A \setminus C. \quad (1)$$

According to our assumption, $Q(\{x_0, y_0\}) \not\subseteq D_{x_0} \bigcup D_{y_0}$, i.e. there exists a point $z \in Q(\{x_0, y_0\}) \bigcap D_{y_1}$, where $y_1 \in A \setminus \{x_0, y_0\}$.

By considering the cases $y_1 \in C$ and $y_1 \in A \setminus C$, we have correspondingly

$$\rho(x_0, y_1) \le \rho(x_0, z) + \rho(z, y_1) < \rho(x_0, z) + \rho(z, y_0) = \rho(x_0, y_0)$$

and

$$\rho(y_0, y_1) \le \rho(y_0, z) + \rho(z, y_1) < \rho(y_0, z) + \rho(z, x_0) = \rho(x_0, y_0),$$

which contradicts to (1). Consequently, $C = A$.

The Lemma is proved.

***Theorem*.** For any strictly monotone function φ the set $A \subset B_k^n$ is $D(\varphi)$-representable code, iff it is constant weight perfect code.

Proof. Sufficiency is obvious.

Necessity. It is enough to consider the case $\varphi(k) = k$ (Lemma 1).

Suppose for the points $x \in A$, $y \in D_x \bigcap B_k^n$, $z \in B_k^n$ we have

$$\rho(x, y) = \rho(x, z). \quad (2)$$

According to Lemma 2, there exists a sequence of points $y = y_1, ..., y_l = z$ in B_k^n satisfying

$$\rho(y_i, y_{i+1}) = 2, \ \rho(x, y_{i+1}) = \rho(x, y), \ i = 1,2,...,l-1. \quad (3)$$

Now suppose for some $i (1 \le i \le l)$ $y_1, y_2, ..., y_{i-1} \in D_x$ and $y_i \notin D_x$. Then there exists a point $z_1 \in A$ such that $y_i \in D_{z_1} \bigcap B_k^n$, i.e.

$$\rho(z_1, y_i) < \rho(x, y_i). \quad (4)$$

As $\rho(y_i, y_{i+1}) = 2$ and $y_{i-1} \in D_x \bigcap B_k^n$, it follows that $\rho(x, y_{i-1}) = \rho(z_1, y_i)$.

From the above-mentioned facts, basing on (3) and (4), we deduce that $\rho(x, y) < \rho(x, y_i)$. But this contradicts the assumption (2). Consequently, if $y \in D_x \bigcap B_k^n$, then any $z \in B_k^n$ point satisfying $\rho(x, y) = \rho(x, z)$ belongs to $D_x \bigcap B_k^n$. It means that all Dirichlet regions of the points of A are spheres in B_k^n.

Suppose x_1 is any point in A. According to Lemma 3, there exists a sequence of points $x_1, x_2 ..., x_l = x$ with $x_i \in M(x_{i-1})$, $i = 1,2...,l$. Consequently, we have $Q(\{x_i, x_{i-1}\}) \bigcap B_k^n \subseteq (D_{x_i} \bigcup D_{x_{i-1}}) \bigcap B_k^n$. Since $D_{x_i} \bigcap B_k^n = S_{t_i}(x_i)$, $D_{x_{i-1}} \bigcap B_k^n = S_{t_{i-1}}(x_{i-1})$, then there exists a point $y_1 \in D_{x_i} \bigcap B_k^n$ such that $\rho(x_i, y_1) = t_i$. But, on the other hand, there exists a point $y_2 \in (Q(\{x_1, x\}) \setminus D_{x_i}) \bigcap B_k^n$ such that $\rho(y_1, y_2) = 2$. It follows from here and the condition $x_i \in M(x_{i-1})$ that $y_2 \in D_{x_{i-1}} \bigcap B_k^n$. Consequently, $\rho(x_{i-1}, y_2) = t_{i-1}$. As a result we have the following system:

$$\begin{cases} \rho(x_i, y_2) \le t_i + 2, \\ \rho(x_{i-1}, y_2) \le t_{i-1}, \end{cases}$$

which is compatible only when $t_i = t_{i-1}$. This means that all Dirichlet regions of the set A are spheres in B_k^n of the same radii. Hence, that set A is a constant weight perfect code.

The Theorem is proved.

The existence of the constant weight perfect codes, besides the trivial ones (i.e. sets, consisting of a single point and sets, consisting of two points with the parameters $n = 4t + 2$, $k = 2t + 1$), is an open question [4]. That is why we would like to present necessary conditions for existence of the constant weight perfect codes [4, 5].

The following conditions are necessary for the existence of the constant weight perfect codes $A \subset B_k^n$ with radius $2t$:

1. $\dfrac{C_n^k}{\sum_{i=0}^{t} C_k^i C_{n-k}^i}$ is an integer;
2. zero of the Eberlein's polynomial $E_t(x) = \sum_{j=0}^{t} (-1)^j C_{x-1}^j C_{k-x}^{t-j} C_{n-k-x}^{t-j}$ are in the set $\{1, 2, \ldots, n\}$;
3. $k \ge t^2 + 3t + 1$;
4. $\dfrac{(2t+1)(k+1)}{t+1} \le n \le \dfrac{(2t+1)(k-1)}{t}$.

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REFERENCES

1. **Zinov'ev V.A. and Leont'ev V.K.** // Problems of Control and Inform. Theory, 1973, v. 2, № 2, p.123–132 (in Russian).
2. **Tiettavainen A.** // SIAM J. Appl. Math., 1973, v. 24, p. 88–96.
3. **Movsisyan G.L., Margaryan Zh.G.** *D*-Representing Code Problem Solution. Fundamentals of Computation Theory. International Conference FCT 87. USSR, Kazan, 1987, p. 328–331.
4. **Delsarte P.** An Algebraic Approach to the Association Schemes of Coding Theory. Philips Research Reports Supplements, 1973, № 10.
5. **Movsisyan G.L.** // Bulletin of MSY, Computing Math. and Cybernetics, 1982, v. 1, № 15, p. 64–69 (in Russian).