PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2013, № 2, p. 8–14

M a t h e m a t i c s

ON BOUNDEDNESS OF A CLASS OF FIRST ORDER LINEAR DIFFERENTIAL OPERATORS IN THE SPACE OF $(n-1)$-DIMENSIONALLY CONTINUOUS FUNCTIONS

V. Zh. DUMANYAN *

Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia

In this article we consider first order linear differential operators in the space of $(n-1)$-dimensionally continuous functions with coefficients having some growth near the domain boundary and prove the boundedness of considered operators.

MSC2010: 30H05; 46E15.

Keywords: bounded differential operator, first order differential operator, $(n-1)$-dimensionally continuous function.

Introduction. Let $Q \subset R_n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial Q \in C^1$. We introduce the following space:

$$\mathcal{U}(Q) \equiv \left\{ u \in W^1_{2,1oc}(Q) \cap C_{n-1}(\bar{Q}) : \quad \int\limits_Q r(x)\, |\nabla u(x)|^2 dx < \infty \right\},$$

$$\|u\|^2_{\mathcal{U}(Q)} = \|u\|^2_{C_{n-1}(\bar{Q})} + \int\limits_Q r(x)\, |\nabla u(x)|^2 dx,$$

where $r(x)$ is the distance from a point $x \in Q$ to the boundary ∂Q and $C_{n-1}(\bar{Q})$ is the Banach space of $(n-1)$-dimensionally continuous functions.

The concept of $(n-1)$-dimensional continuity was introduced by Gushchin [1] and is as follows.

Let μ be a nonnegative unit measure on R_n, with support in $\bar{Q}$, and satisfying the following condition: there exists a constant $C = C(\mu)$ such that for all $r > 0$ and $x^0 \in \bar{Q}$ the measure of the ball $B_r(x^0)$ of centre x^0 and radius r is majorized by Cr^{n-1}, i.e.

$$\mu(B_r(x^0)) \leq Cr^{n-1} \quad \text{for all} \quad r > 0 \quad \text{and} \quad x^0 \in \bar{Q} \tag{1}$$

* E-mail: duman@ysu.am

the smallest such C is called the norm of μ, written $\|\mu\|$. The Gushchin space of $(n-1)$-dimensionally continuous functions $C_{n-1}(\bar{Q})$ is the completion of the space of continuous functions on $\bar{Q}$ with respect to the norm

$$\|u\|_{C_{n-1}(\bar{Q})} = \sup \left(\frac{1}{\|\mu\|} \int\limits_{\bar{Q}} u^2(x)\, d\mu(x) \right)^{1/2},$$

where the supremum is taken over all measures μ satisfying (1). Functions in $C_{n-1}(\bar{Q}) \subset L_2(Q)$ have traces on sets of positive $(n-1)$-dimensional Hausdorff measure, and the set of traces of all functions from $C_{n-1}(\bar{Q})$ on the smooth $(n-1)$-dimensional surface $\Gamma \subset \bar{Q}$ coincides with $L_2(\Gamma)$.

There is an equivalent definition of $(n-1)$-dimensional continuity in terms of the proximity of the values of functions on neighbouring measures.

Let μ, ν be nonnegative unit measures on R_n, with support in $\bar{Q}$, satisfying (1), and let ϕ be a measure on R_{2n}, with support in $\bar{Q} \times \bar{Q}$. We also assume that $\mu(G) = \phi(G \times R_n), \nu(G) = \phi(R_n \times G)$ for all (Borel) sets $G \subset \bar{Q}$.

A function u is said to be $(n-1)$-dimensionally continuous, if for any positive ε there exists a positive δ such that

$$\frac{1}{\|\mu\| + \|\nu\|} \int\limits_{\bar{Q} \times \bar{Q}} \Big(u(x) - u(y) \Big)^2 d\phi(x,y) < \varepsilon$$

(the distance between the values of u on the measures μ and ν along ϕ is less than ε), provided

$$\int\limits_{\bar{Q} \times \bar{Q}} |x - y|\, d\phi(x,y) < \delta$$

(the distance between the measures μ and ν along ϕ is less than δ); see [1] for more details.

Note that the space $\mathfrak{U}(Q)$ was introduced in [1], in the paper devoted to the investigation of Dirichlet problem for second order elliptic equation without lower order terms. The study of Dirichlet problem for general second order elliptic equation requires, in particular, investigation of properties of the first order differential operator

$$Tu \equiv -(\bar{b}, \nabla u) + \mathrm{div}(\bar{c}u) - du,\ u \in \mathfrak{U}(Q),$$

where the coefficients $\bar{b}(x) = (b_1(x), \ldots, b_n(x))$, $\bar{c}(x) = (c_1(x), \ldots, c_n(x))$ and $d(x)$ are measurable and bounded on any strictly interior subdomain of Q.

For an arbitrary $u \in \mathfrak{U}(Q)$, define a linear functional Tu acting on $\overset{\circ}{W}{}_2^1(Q)$ as

$$\langle Tu, v \rangle \equiv -\int\limits_{Q} ((\bar{b}(x), \nabla u(x))v(x) - (\bar{c}(x)u(x), \nabla v(x)) -$$

$$-d(x)u(x)v(x))dx,\ v \in \overset{\circ}{W}{}_2^1(Q).$$

The main result of the present paper is the following

Theorem. Suppose

$$\int\limits_0 B^2(t)\,dt < \infty, \quad \text{where } B(t) \equiv \sup_{r(x)\geq t} |\bar{b}(x)|, \tag{2}$$

$$\int\limits_0 C^2(t)\,dt < \infty, \quad \text{where } C(t) \equiv \sup_{r(x)\geq t} |\bar{c}(x)|, \tag{3}$$

$$\int\limits_0 t^2 D^2(t)\,dt < \infty, \quad \text{where } D(t) \equiv \sup_{r(x)\geq t} |d(x)|. \tag{4}$$

Then T is a bounded linear operator from $\mathfrak{U}(Q)$ to $\overset{\circ}{W}{}_2^{-1}(Q)$.

Proof. Let $x^0 \in \partial Q$ be an arbitrary boundary point of the domain Q. We fix a local coordinate system (x', x_n) with origin at x^0, and take the x_n-axis along the inward normal vector $\nu(x^0)$ to ∂Q at x^0. The boundary ∂Q being C^1-smooth, there exists a number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R_{n-1})$,

$$\varphi_{x^0}(0) = 0,\ \nabla\varphi_{x^0}(0) = 0,\ |\nabla\varphi_{x^0}(x')| \leq \frac{1}{2} \ \text{ for all } x' \in R_{n-1},$$

such that the intersection of Q with the open ball $U_{x^0}^{(r_{x^0})} = \left\{x : |x - x^0| < r_{x^0}\right\}$ with centre x^0 and radius r_{x^0}, has the representation

$$Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \left\{(x', x_n) : x_n > \varphi_{x^0}(x')\right\}.$$

Hence,

$$\partial Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \left\{(x', x_n) : x_n = \varphi_{x^0}(x')\right\}.$$

Let $\ell_{x^0} = \dfrac{r_{x^0}}{\sqrt{2}}$. Choose $U_{x^m}^{(\ell_{x^m})}, m = 1, \ldots, p$, to be a finite open refinement of the open cover $\left\{U_{x^0}^{(\ell_{x^0})}, x^0 \in \partial Q\right\}$ of the boundary ∂Q. Also let

$$U_m = U_{x^m}^{(r_{x^m})},\ r_m = r_{x^m},\ \ell_m = \ell_{x^m},\ \varphi_m = \varphi_{x^m}, \text{ where } m = 1, \ldots, p.$$

We set $h = \dfrac{1}{3}\left(\dfrac{2}{\sqrt{5}} - \dfrac{\sqrt{2}}{2}\right)\min(r_1, \ldots, r_p)$. Then each of the "curvilinear cylinders"

$$\Pi_m^{\ell_m, h} = \left\{(x', x_n) : |x'| < \ell_m, \varphi_m(x') < x_n < \varphi_m(x') + h\right\}, m = 1, \ldots, p,$$

lies in the corresponding ball U_m and hence in $U_m \cap Q$.

Let $\ell_0 < h$ be a positive number such that the complement in Q of the domain

$$Q_{\ell_0} = \left\{x \in Q : r(x) = \text{ dist}\,(x, \partial Q) > \ell_0\right\}$$

is contained in the union of the "cylinders" $\Pi_m^{\ell_m, h}$, $m = 1, \ldots, p$, that is,

$$Q^{\ell_0} = \left\{x \in Q : r(x) = \text{ dist}\,(x, \partial Q) \leq \ell_0\right\} \subset \bigcup_{m=1}^{p} \Pi^{\ell_m, h}.$$

It can be easily seen that for any $x=(x',x_n)\in\Pi_m^{\ell_m,h}, m=1,\dots,p$,

$$r(x)\le x_n-\varphi_m(x')\le\frac{\sqrt{5}}{2}r(x).$$

We fix some m, $1\le m\le p$, and consider a local coordinate system with origin at x^m. In what follows, we suppress the dependence of the function φ_m on m, writing for brevity $\varphi=\varphi_m$.

We define the mappings $\mathcal{L}$ and $\mathcal{L}_{-1}$ of the space R_n onto itself by

$$\mathcal{L}(x)=(x',x_n-\varphi(x')),\ x=(x',x_n),$$
$$\mathcal{L}_{-1}(y)=(y',y_n+\varphi(y')),\ y=(y',y_n),$$

respectively. Let $\tilde{\Pi}_m^{\ell_m,h}=\mathcal{L}(\Pi_m^{\ell_m,h})$ be the image of $\Pi_m^{\ell_m,h}$ under $\mathcal{L}$.

Given arbitrary $u\in\mathfrak{U}(Q)$ and $\eta\in C_0^\infty(Q)$, we set

$$u(y',y_n+\varphi(y'))=\tilde{u}(y),\quad \eta(y',y_n+\varphi(y'))=\tilde{\eta}(y).$$

Further, consider

$$\langle Tu,\eta\rangle\equiv-\int\limits_Q(\bar{b}(x),\nabla u(x))\eta(x)dx-\int\limits_Q(\bar{c}(x)u(x),\nabla\eta(x))dx-\int\limits_Q d(x)u(x)\eta(x)dx.$$

First, it is readily verified that $f(t)=o\left(\frac{1}{t}\right)$ at $t\to+0$ for a monotonously decreasing function $f(t)$, $t\ge0$, having the finite integral $\int\limits_0 f(t)\,dt<\infty$.

Then, using (2), we obtain $B(t)\le\dfrac{\text{const}}{t^{1/2}}$ and

$$\left|\int\limits_Q(\bar{b}(x),\nabla u(x))\eta(x)\,dx\right|\le\int\limits_Q|\bar{b}(x)||\nabla u(x)||\eta(x)|\,dx\le$$

$$\le\left(\int\limits_Q r(x)\,|\nabla u(x)|^2\,dx\right)^{1/2}\left(\int\limits_Q\frac{B^2(r(x))}{r(x)}\eta^2(x)\,dx\right)^{1/2}\le$$

$$\le\text{const}\,\|u\|_{\mathfrak{U}(Q)}\left(\int\limits_Q\frac{\eta^2(x)}{r^2(x)}\,dx\right)^{1/2}.$$

Further, by Hardy's inequality [4],

$$\int\limits_Q\frac{\eta^2(x)}{r^2(x)}\,dx\le\int\limits_{Q_{l_0}}\frac{\eta^2(x)}{r^2(x)}\,dx+\sum_{m=1}^{p}\int\limits_{\Pi_m^{\ell_m,h}}\frac{\eta^2(x)}{r^2(x)}\,dx\le$$

$$\le\frac{1}{\ell_0^2}\int\limits_Q\eta^2(x)\,dx+\sum_{m=1}^{p}\left(\frac{\sqrt{5}}{2}\right)^2\int\limits_{\tilde{\Pi}_m^{\ell_m,h}}\frac{\tilde{\eta}^2(y)}{y_n^2}\,dy=\frac{1}{\ell_0^2}\int\limits_Q\eta^2(x)\,dx+$$

$$+\sum_{m=1}^{p}\left(\frac{\sqrt{5}}{2}\right)^{2}\int\limits_{0}^{h}dy_n\int\limits_{|y'|<\ell_m}\frac{\left(\int\limits_{o}^{y_n}\tilde{\eta}_\tau(y',\tau)\,d\tau\right)^{2}}{y_n^2}dy'\leq \text{const}\int\limits_{Q}|\nabla\eta|^2\,dx+$$

$$+\sum_{m=1}^{p}\left(\frac{\sqrt{5}}{2}\right)^{2}\int\limits_{|y'|<\ell_m}dy'\int\limits_{0}^{h}\left(\int\limits_{o}^{y_n}\tilde{\eta}_\tau(y',\tau)\,d\tau/y_n\right)^{2}dy_n\leq$$

$$\leq \text{const}\left(\int\limits_{Q}|\nabla\eta|^2\,dx+\sum_{m=1}^{p}\int\limits_{|y'|<\ell_m}dy'\int\limits_{0}^{h}\tilde{\eta}_{y_n}^2(y',y_n)\,dy_n\right)\leq$$

$$\leq \text{const}\left(\int\limits_{Q}|\nabla\eta|^2\,dx+\sum_{m=1}^{p}\int\limits_{\tilde{\Pi}_m^{\ell_m,h}}|\nabla\tilde{\eta}(y)|^2\,dy\right)\leq \text{const}\int\limits_{Q}|\nabla\eta(x)|^2\,dx. \tag{5}$$

Hence,

$$\left|\int\limits_{Q}(\bar{b}(x),\nabla u(x))\eta(x)\,dx\right|\leq \text{const}\,\|u\|_{\mathfrak{U}(Q)}\left(\int\limits_{Q}|\nabla\eta(x)|^2\,dx\right)^{1/2}\leq$$

$$\leq \text{const}\,\|u\|_{\mathfrak{U}(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}, \tag{6}$$

where the constant is independent of u and η.

Using (3), we obtain

$$\left|\int\limits_{Q}(\bar{c}(x)u(x),\nabla\eta(x))dx\right|\leq\int\limits_{Q}C(r(x))|u(x)||\nabla\eta(x)|dx\leq$$

$$\leq\left(\int\limits_{Q}C^2(r(x))u^2(x)\,dx\cdot\int\limits_{Q}|\nabla\eta(x)|^2\,dx\right)^{1/2}\leq$$

$$\leq \text{const}\left(\int\limits_{Q}C^2(r(x))u^2(x)dx\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\leq$$

$$\leq \text{const}\left(C^2(\ell_0)\int\limits_{Q_{\ell_0}}u^2(x)\,dx+\sum_{m=1}^{p}\int\limits_{\Pi_m^{\ell_m,h}}C^2(r(x))u^2(x)dx\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\leq$$

$$\leq \text{const}\left(\|u\|_{L_2(Q)}^2+\sum_{m=1}^{p}\int\limits_{\tilde{\Pi}_m^{\ell_m,h}}C^2\left(\frac{2}{\sqrt{5}}y_n\right)\tilde{u}^2(y',y_n)dy'dy_n\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\leq$$

$$\le \mathrm{const} \left(\|u\|^2_{L_2(Q)} + \sum_{m=1}^{p} \int_0^h C^2 \left(\frac{2}{\sqrt{5}} y_n \right) dy_n \max_{0 \le y_n \le h} \int_{|y'|<\ell_m} \tilde{u}^2(y', y_n) dy' \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le$$

$$\le \mathrm{const} \left(\|u\|^2_{L_2(Q)} + \sum_{m=1}^{p} \max_{0 \le y_n \le h} \int_{|y'|<\ell_m} \tilde{u}^2(y', y_n) dy' \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le$$

$$\le \mathrm{const} \left(\|u\|^2_{L_2(Q)} + \|u\|^2_{C_{n-1}(\bar{Q})} \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)}.$$

Further, it follows by $C_{n-1}(\bar{Q}) \subset L_2(Q)$ that

$$\left| \int_Q (\bar{c}(x)u(x), \nabla\eta(x)) dx \right| \le \mathrm{const} \|u\|_{C_{n-1}(\bar{Q})} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le$$

$$\le \mathrm{const} \|u\|_{\mathfrak{U}(Q)} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)}, \tag{7}$$

where the constant is independent of u and η.

Applying (4) and (5), we have, by analogy with the previous estimates,

$$\left| \int_Q d(x)u(x)\eta(x)\, dx \right| \le \int_Q D(r(x))|u(x)||\eta(x)|\, dx \le$$

$$\le \left(\int_Q r^2(x) D^2(r(x)) u^2(x)\, dx \cdot \int_Q \frac{\eta^2(x)}{r^2(x)}\, dx \right)^{1/2} \le$$

$$\le \mathrm{const} \left(\int_Q r^2(x) D^2(r(x)) u^2(x)\, dx \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le$$

$$\le \mathrm{const} \left(D^2(\ell_0) \int_{Q_{\ell_0}} r^2(x) u^2(x)\, dx + \sum_{m=1}^{p} \int_{\Pi_m^{\ell_m,h}} r^2(x) D^2(r(x)) u^2(x)\, dx \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le$$

$$\le \mathrm{const} \left(D^2(\ell_0) \max_{x \in \bar{Q}_{\ell_0}} r^2(x) \int_{Q_{\ell_0}} u^2(x)\, dx + \right.$$

$$\left. + \sum_{m=1}^{p} \int_{\tilde{\Pi}_m^{\ell_m,h}} y_n^2 D^2 \left(\frac{2}{\sqrt{5}} y_n \right) \tilde{u}^2(y', y_n)\, dy'\, dy_n \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}^1_2(Q)} \le \mathrm{const} \left(\|u\|^2_{L_2(Q)} + \right.$$

$$+\sum_{m=1}^{p}\int_0^h y_n^2\,D^2\left(\frac{2}{\sqrt{5}}y_n\right)dy_n \max_{0\le y_n\le h}\int_{|y'|<\ell_m}\tilde{u}^2(y',y_n)\,dy'\Bigg)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\le$$

$$\le \text{const}\left(\|u\|^2_{L_2(Q)}+\sum_{m=1}^{p}\max_{0\le y_n\le h}\int_{|} y'|<\ell_m\tilde{u}^2(y',y_n)dy'\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\le$$

$$\le \text{const}\left(\|u\|^2_{L_2(Q)}+\|u\|^2_{C_{n-1}(\bar{Q})}\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\le \text{const}\|u\|_{\mathcal{U}(Q)}\,\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}, \qquad (8)$$

where the constant is independent of u and η.

Consequently, it follows from (6), (7) and (8) that for arbitrary $u\in\mathcal{U}(Q)$ and $\eta\in C_0^\infty(Q)$

$$|\langle Tu,\eta\rangle|\le \text{const}\|u\|_{\mathcal{U}(Q)}\,\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)},$$

where the constant is independent of u and η. Since the functions η of $C_0^\infty(Q)$ form a dense subset of $\overset{\circ}{W}{}_2^1(Q)$, the estimate just obtained forces the operator

$$T:\mathcal{U}(Q)\to \overset{\circ}{W}{}_2^{-1}(Q)$$

to be bounded. □

Remark. It is clear, that condition (2) of the Theorem can be replaced by the following:

$$|\bar{b}(x)|\le\frac{\text{const}}{r^{1/2}(x)},\quad x\in Q.$$

Received 03.05.2013

REFERENCES

1. **Gushchin A.K.** On the Dirichlet Problem for a Second-Order Elliptic Equation. // Mat. Sb. (N.S.), 1988, v. 137(179), № 1(9), p. 19–64.
2. **Dumanyan V.Zh.** On Solvability of the Dirichlet Problem for Elliptic Equations. // Izvestia NAN Armenii. Matematika, 2011, v. 46, № 2, p. 37–52 (in Russian).
3. **Dumanyan V.Zh.** Solvability of the Dirichlet Problem for a General Second-Order Elliptic Equation. // Mat. Sb., 2011, v. 202, № 7, p. 75–94 (in Russian).
4. **Maz'ya V.G.** Sobolev Spaces. L.: LGU, 1985, 416 p. (in Russian).