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M a t h e m a t i c s

THE AUTOMORPHISM TOWER PROBLEM FOR FREE PERIODIC GROUPS

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We prove that the group of automorphisms $Aut(B(m,n))$ of the free Burnside group $B(m,n)$ is complete for every odd exponent $n \geq 1003$ and for any $m > 1$, that is it has a trivial center and any automorphism of $Aut(B(m,n))$ is inner. Thus, the automorphism tower problem for groups $B(m,n)$ is solved and it is showed that it is as short as the automorphism tower of the absolutely free groups. Moreover, we obtain that the group of all inner automorphisms $Inn(B(m,n))$ is the unique normal subgroup in $Aut(B(m,n))$ among all its subgroups, which are isomorphic to free Burnside group $B(s,n)$ of some rank s.

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Introduction. If the center of a group G is trivial, then it is embedded into the group of its automorphisms $Aut(G)$. Such embedding is given by mapping each element of group to the inner automorphism generated by this element. The inner automorphism generated by an element $g \in G$ is denoted by i_g and is defined by the formula $x^{i_g} = gxg^{-1}$ for all $x \in G$ (the image of the element x under the map α is denoted by x^{α}). The easily verifiable relation for composition of automorphisms $\alpha \circ i_g \circ \alpha^{-1} = i_{g^{\alpha}}$ shows that the group of all inner automorphisms $Inn(G)$ is a normal subgroup in $Aut(G)$. Moreover, the relation $\alpha \circ i_g \circ \alpha^{-1} = i_{g^{\alpha}}$ implies that in a group G with trivial center the centralizer of the subgroup $Inn(G)$ is also trivial in $Aut(G)$. In particular, the group $Aut(G)$ is also a centerless group. This allows to consider the automorphism tower

$$G = G_0 \lhd G_1 \lhd \cdots \lhd G_k \lhd \cdots, \tag{1}$$

where $G_k = Aut(G_{k-1})$ and G_k is identified with $Inn(G_k)$ under the embedding $G_k \hookrightarrow Aut(G_k)$, $g \mapsto i_g$ $(k = 1, 2, ...)$.

According to classical Wielandt's theorem (see [1], Theorem 13.5.2), the automorphism tower (1) of any finite centerless group terminates after a finite number

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of steps (that is there exists a number k such that $G_k = G_{k-1}$). For infinite groups the analogous statement is false (for example, automorphism tower of infinite dihedral group does not terminate in finitely many steps). In early seventies G. Baumslag proposed to study the automorphism tower for absolutely free and for some relatively free groups. In particular, he formulated a hypothesis that the tower of absolutely free group of finite rank should be "very short".

In 1975 J. Dyer and E. Formanek in [2] confirmed the Baumsalag's hypothesis proving, that if F is a free group of finite rank > 1, then its group of automorphisms $Aut(F)$ is complete. Recall that a group is called *complete*, if it is centerless and each of its automorphisms is inner. V. Tolstikh in [3] proved the completeness of $Aut(F)$ for free groups F of infinite rank. It is clear that if the group of automorphisms of centerless group G_0 is complete, then its automorphism tower terminates after the first step, that is, $G_0 \lhd G_1 = G_2 = \dots$.

The later new proofs and various generalizations of Dayer-Formanek theorem have been obtained by E. Formanek [4], D.G. Khramtsov [5], M.R. Bridson and K. Vogtmann [6].

Further, in [7] and [8] it was established that the group of automorphisms of each non-abelian free solvable group of finite rank is complete. It was showed that the group of automorphisms of free nilpotent group of class 2 and rank $r \geq 2$ is complete provided that $r \neq 3$. In the case n = 3 the height of the automorphism tower (1) is 2.

Note that in all above-mentioned results on automorphism tower of relatively free groups only torsion free groups were considered.

Preliminary and Main Results. We study the automorphism tower of free Burnside groups $B(m,n)$, i.e. the relatively free groups of rank $m > 1$ of the variety of all groups, which satisfy the identity $x^n = 1$. The group $B(m,n)$ is the quotient group of absolutely free group F_m on m generators by normal subgroup F_m^n generated by all n-th powers. Obviously, any periodic group of exponent n with m generators is a quotient group of $B(m,n)$. By the theorem of S.I. Adian, the group $B(m,n)$ of rank $m > 1$ is infinite for any odd $n \geq 665$. This Theorem and a series of fundamental properties of $B(m,n)$ was proved in the monograph [9]. A comprehensive survey of results about the free Burnside groups and related topics is given in [10].

Our main result states that the automorphism tower of non-cyclic free Burnside group $B(m,n)$ is terminated on the first step for any odd $n \geq 1003$. Hence, the automorphism tower problem for groups $B(m,n)$ is solved. We is show that it is as short as the automorphism tower of the absolutely free groups. In particular, the group $Aut(B(m,n))$ is complete. The inequality $n \geq 1003$ for the exponent n is closely related to the result in the last chapter of [9], concerning the construction of an infinite independent system of group identities in two variables (the solution of the finite basis problem). Based on the developed technique in this chapter in [11], the authors have constructed an infinite simple group of period n with cyclic subgroups for each $n \geq 1003$, which plays a key role in our proof of the main result. The use of simple quotient groups to obtain information about the automorphisms of $B(m,n)$ first occurs in the paper [12] of A.Yu. Olshanskii. We are pleased to stress the influence of [12] on our research.

The well known Gelder-Bear's theorem asserts that every complete group is a direct factor in any group, in which it is contained as a normal subgroup (see [1], Theorem 13.5.7). According to Adian's theorem (see [9], Theorem 3.4) for any odd $n \geq 665$ the center of (non-cyclic) free Burnside group is trivial. However, the groups $B(m,n)$ are not complete, because, for example, the automorphism ϕ of $B(m,n)$, defined on the free generators by the formula $\forall i(\phi(a_i) = a_i^2)$, is an outer automorphism.

Nevertheless, the free Burnside groups possess a property analogous to the above-mentioned characteristic property of complete groups. It turns out that each group $B(m,n)$ is a direct factor in every periodic group of exponent n, in which it is contained as a normal subgroup. This statement was proved for large enough odd n ($n > 10^{80}$) by E. Cherepanov in [13] and for all odd $n \geq 1003$ by the author in [14].

Our main result is the following

Theorem 1. For any odd $n \geq 1003$ and $m > 1$, the group of all inner automorphisms $Inn(B(m,n))$ is the unique normal subgroup of the group $Aut(B(m,n))$ among all subgroups, which are isomorphic to a free Burnside group $B(s,n)$ of some rank s.

The proof of this result is essentially based on the papers [15–17] of the author.

Theorem 1 immediately implies the following

Theorem 2. The groups of automorphisms $Aut(B(m,n))$ and $Aut(B(k,n))$ are isomorphic, if and only if $m = k$ (for any odd $n \geq 1003$).

By Burnside criterion, if the group of all inner automorphisms $Inn(G)$ of a centerless group G is a characteristic subgroup in $Aut(G)$, then $Aut(G)$ is complete (see [1], Theorem 13.5.8). Since the image of the subgroup $Inn(B(m,n))$ under every automorphism of the group $Aut(B(m,n))$ is a normal subgroup, Theorem 1 implies

Theorem 3. The group of automorphisms $Aut(B(m,n))$ of the free Burnside group $B(m,n)$ is complete for any odd $n \geq 1003$ and $m > 1$.

Theorem 3 in a some sense gives a solution of the problem *8.53 a)* formulated by A.Yu. Olshanskii in the Kourovka Notebook [18]: *Let n be large enough odd number. Describe the automorphisms of free Burnside group $B(m,n)$ of exponent n with m generators.*

It should be emphasized that the group $Aut(B(m,n))$ is saturated with a lot of subgroups, which are isomorphic to some free Burnside group. It is known that each non-cyclic subgroup of $B(m,n)$ and, hence, the group $Inn(B(m,n))$ contains a subgroup isomorphic to the free Burnside group $B(\infty,n)$ of infinite rank (see [19], Theorem 1). Furthermore, $Aut(B(m,n))$ contains periodic subgroups having trivial intersection with $Inn(B(m,n))$ for $m > 2$. For instance, consider a subgroup of $Aut(B(m,n))$ generated by automorphisms l_j, $j = 2,...,m$, defined on generators a_i, $i = 1,...,m$, by formulae $l_j(a_1) = a_1 a_j$ and $l_j(a_k) = a_k$ for $k = 2,...,m$. It is easy to check the equality

$$W(l_2,...,l_m)(a_1) = a_1 \cdot W(a_2,...,a_m)$$

for any word $W(a_2,...,a_m)$. Then, the automorphism $W(l_2,...,l_m)$ is the identity automorphism, if and only if the equality $W(a_2,...,a_m) = 1$ holds in $B(m,n)$. Hence, the automorphisms l_{1j}, $j = 2,...,m$, generate a subgroup isomorphic to free Burnside group $B(m-1,n)$ of rank $m-1$. According to Theorem 1, none of the above-mentioned subgroups is a normal subgroup of $Aut(B(m,n))$.

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