PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2016, № 1, p. 12–21

Mathematics

MOORE–PENROSE INVERSE OF BIDIAGONAL MATRICES. III

Yu. R. HAKOPIAN[1]*, S. S. ALEKSANYAN[2]**

[1] *Chair of Numerical Analysis and Mathematical Modeling YSU, Armenia*
[2] *Chair of Mathematics and Mathematical Modeling RAU, Armenia*

The present paper is a direct continuation of the papers [1, 2]. We obtain intermediate results, which will be used in the next final fourth part of this study, where a definitive solution to the Moore–Penrose inversion problem for singular upper bidiagonal matrices is given.

MSC2010: Primary 15A09; Secondary 65F20.

***Keywords*:** generalized inverse, Moore–Penrose inverse, bidiagonal matrix.

Introduction. In this paper we continue the study started in our previous papers [1, 2]. In [2] we consider the Moore–Penrose inversion problem for singular upper bidiagonal matrices

$$A = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \qquad (1)$$

with any arrangement of one or more zeros on the main diagonal and under assumption $b_1, b_2, \ldots, b_{n-1} \neq 0$. To solve the problem in [2] we carried out some preliminary constructions and calculations. Recall the main steps of the proposed approach. First we have represented the matrix (1) in the block form

$$A = \begin{bmatrix} A_1 & B_1 & & & \\ & A_2 & B_2 & & \\ & & \ddots & \ddots & \\ & & & A_{m-1} & B_{m-1} \\ & & & & A_m \end{bmatrix} \qquad (2)$$

with diagonal blocks A_k, $k = 1, 2, \ldots, m$, of the size $n_k \times n_k$ and over-diagonal blocks B_k, $k = 1, 2, \ldots, m-1$, of the size $n_k \times n_{k+1}$, where $n_1 + n_2 + \cdots + n_m = n$.

* E-mail: yuri.hakopian@ysu.am ** E-mail: alexosarm@gmail.com

The structure of the blocks was specified in the **Introduction** of [2]. Further, it has been shown that the Moore–Penrose inverse A^+ of the matrix (2) has the following block form:

$$A^+ = \begin{bmatrix} Z_1 & & & & \\ H_2 & Z_2 & & 0 & \\ & \ddots & \ddots & & \\ & 0 & H_{m-1} & Z_{m-1} & \\ & & & H_m & Z_m \end{bmatrix}, \tag{3}$$

and the blocks Z_k and H_k are computed by the formulae

$$Z_k = \lim_{\varepsilon\to+0} L_k(\varepsilon)^{-1} A_k^T, \quad k = 1,2,\ldots,m, \tag{4}$$

$$H_k = \lim_{\varepsilon\to+0} L_k(\varepsilon)^{-1} B_{k-1}^T, \quad k = 2,3,\ldots,m, \tag{5}$$

where

$$L_1(\varepsilon) = A_1^T A_1 + \varepsilon I_1, \tag{6}$$

$$L_k(\varepsilon) = A_k^T A_k + B_{k-1}^T B_{k-1} + \varepsilon I_k, \quad k = 2,3,\ldots,m, \tag{7}$$

and I_k stands for the identity matrix of order n_k (see [2] **A Way of Computing the Moore–Penrose Invertion**).

The problem of computing the block Z_1 is completely discussed in [2] (see **Block $\mathbf{Z_1}$**).

As it is seen from (4) and (7) for the values $k = 2,3,\ldots,m$, the blocks Z_k are computed by similar formulae. The same can be said about the blocks H_k (see (5) and (7)). The difference consists only in sizes and types of the diagonal blocks A_k (see [2] **Introduction**). Note that each block A_k separately is an upper bidiagonal matrix. Therefore we realize the computation of the blocks Z_k and H_k by solving several standard model problems. To this end we introduced a model tridiagonal matrix

$$L(\varepsilon) = A^T A + B^T B + \varepsilon I, \tag{8}$$

which is constructed by other model matrices

$$A = \begin{bmatrix} d_1 & b_1 & & & \\ & d_2 & b_2 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & d_{n-1} & b_{n-1} \\ & & & & d_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \\ \Delta & 0 & \ldots & 0 \end{bmatrix} \tag{9}$$

(we assume that the entries $b_1, b_2, \ldots, b_{n-1}$ of the matrix A as well as the entry Δ of the $l \times n$ matrix B are nonzero.)

In the model problems we will also consider the case when the matrix A from (9) is nonsingular. The formulae for the entries of the inverse matrix $L(\varepsilon)^{-1}$ were derived in [2] (see **Invertion of a Model Matrix $L(\varepsilon)$**).

Thus, as it was mentioned in [2], our next task is to compute the model matrices

$$Z=\lim_{\varepsilon\to+0}L(\varepsilon)^{-1}A^T \tag{10}$$

and

$$H=\lim_{\varepsilon\to+0}L(\varepsilon)^{-1}B^T, \tag{11}$$

where $L(\varepsilon)$, A, B are specified in (8) and (9). In this connection we will separately consider the cases A, B and C outlined in [2] (**Invertion of a Model Matrix** $L(\varepsilon)$).

Computation of the Model Matrices Z and H.

Case A. Remind that this case provides $n\geq 1$ and $d_1,d_2,\dots,d_n\neq 0$.

If $n=1$, then the matrices Z and H have a simple view:

$$Z=\left[\frac{d_1}{d_1^2+\Delta^2}\right]_{1\times 1},\quad H=\left[0\dots 0\,\frac{\Delta}{d_1^2+\Delta^2}\right]_{1\times l}. \tag{12}$$

It can be easily obtained from the equalities (10) and (11).

The case $n\geq 2$ is more complicated. Let us start with the computation of the matrix $Z=[z_{ij}]_{n\times n}$. For this purpose consider the matrix

$$L(\varepsilon)^{-1}A^T\equiv Y(\varepsilon)=[y_{ij}(\varepsilon)]_{n\times n}. \tag{13}$$

According to the definition (10) of the matrix Z, we have

$$z_{ij}=\lim_{\varepsilon\to+0}y_{ij}(\varepsilon),\quad i,j=1,2,\dots,n. \tag{14}$$

As it follows from (13), for indeces $1\leq j\leq n-1$ the entries $y_{ij}(\varepsilon)$ of the matrix $Y(\varepsilon)$ are calculated by the rule

$$y_{ij}(\varepsilon)=x_{ij}d_j+x_{i\,j+1}b_j,\quad i=1,2,\dots,n. \tag{15}$$

For a fixed index j in the range $1\leq j\leq n-1$ let us separately consider the cases $i=1,2,\dots,j$ and $i=j+1,j+2,\dots,n$.

• ***Case*** $i=1,2,\dots,j$.

Using the expressions for the entries x_{ij} of the matrix $L(\varepsilon)^{-1}$ (see formula (43) in [2]), from (15) we obtain

$$y_{ij}(\varepsilon)=t\nu_i(\mu_jd_j+\mu_{j+1}b_j). \tag{16}$$

Then, using the representations of the quantities μ_i (see (27) in [2]), we get

$$\mu_jd_j+\mu_{j+1}b_j=(\mathring{\mu}_j\,d_j+\mathring{\mu}_{j+1}\,b_j)+O(\varepsilon). \tag{17}$$

Substituting the expression (17) as well as the representations of the quantities ν_i and t (see (33) and (39) in [2]) into the right hand side of the equality (16), we obtain

$$y_{ij}(\varepsilon)=\frac{(\mathring{\nu}_i+O(\varepsilon))((\mathring{\mu}_j\,d_j+\mathring{\mu}_{j+1}\,b_j)+O(\varepsilon))}{\mathring{t}+O(\varepsilon)}.$$

Then, calculating the limit as $\varepsilon\to+0$, according to (14), we find that

$$z_{ij}=\frac{\mathring{\nu}_i\,(\mathring{\mu}_j\,d_j+\mathring{\mu}_{j+1}\,b_j)}{\mathring{t}},\quad i=1,2,\dots,j. \tag{18}$$

In the [2](**Case B**) we obtained the expression (40) for the quantity $\overset{\circ}{t}$. Therefore, we get

$$z_{ij} = \frac{\overset{\circ}{\nu}_i \,(\overset{\circ}{\mu}_j\, d_j + \overset{\circ}{\mu}_{j+1}\, b_j)}{d_n^2\, \overset{\circ}{\nu}_n + \Delta^2 \alpha_1}, \quad i = 1,2,\dots,j. \tag{19}$$

To derive the closed form expressions for the entries of the matrix Z, let us trasform the right hand side of the equality (19). First consider the sum $\overset{\circ}{\mu}_j\, d_j + \overset{\circ}{\mu}_{j+1}\, b_j$. Using the closed form expression for the quantity $\overset{\circ}{\mu}_i$ (see formula (29) in [2]), we have

$$\begin{aligned}
\overset{\circ}{\mu}_j\, d_j + \overset{\circ}{\mu}_{j+1}\, b_j &= (-1)^{n-j}\left[d_j \prod_{s=j}^{n-1} r_s - b_j \prod_{s=j+1}^{n-1} r_s\right] \\
&\quad + (-1)^{n-j} d_n^2 \left[d_j \sum_{k=j}^{n-1} \frac{1}{d_k^2}\left(\prod_{s=j}^{k-1} r_s\right)\left(\prod_{s=k}^{n-1} \frac{1}{r_s}\right)\right. \\
&\quad \left. - b_j \sum_{k=j+1}^{n-1} \frac{1}{d_k^2}\left(\prod_{s=j+1}^{k-1} r_s\right)\left(\prod_{s=k}^{n-1} \frac{1}{r_s}\right)\right] \equiv J_1 + J_2\,.
\end{aligned} \tag{20}$$

It can be readily shown that $J_1 = 0$. The addend J_2 is transformed as follows:

$$\begin{aligned}
J_2 &= (-1)^{n-j} d_n^2 d_j \left[\sum_{k=j}^{n-1} \frac{1}{d_k^2}\left(\prod_{s=j}^{k-1} r_s\right)\left(\prod_{s=k}^{n-1} \frac{1}{r_s}\right)\right. \\
&\quad \left. - \sum_{k=j+1}^{n-1} \frac{1}{d_k^2}\left(\prod_{s=j}^{k-1} r_s\right)\left(\prod_{s=k}^{n-1} \frac{1}{r_s}\right)\right] \\
&= (-1)^{n-j} d_n^2 d_j \cdot \frac{1}{d_j^2} \prod_{s=j}^{n-1} \frac{1}{r_s} = (-1)^{n-j} \frac{d_n^2}{d_j} \prod_{s=j}^{n-1} \frac{1}{r_s}\,.
\end{aligned} \tag{21}$$

Thus, from (20) and (21) we get

$$\overset{\circ}{\mu}_j\, d_j + \overset{\circ}{\mu}_{j+1}\, b_j = (-1)^{n-j} \frac{d_n^2}{d_j} \prod_{s=j}^{n-1} \frac{1}{r_s}\,. \tag{22}$$

Further, using the expressions for the quantites α_i and $\overset{\circ}{\nu}_i$ (see formulae (31) and (35) from [2]), we can write the sum $d_n^2\, \overset{\circ}{\nu}_n + \Delta^2 \alpha_1$ in the following form:

$$\begin{aligned}
&d_n^2\, \overset{\circ}{\nu}_n + \Delta^2 \alpha_1 = \\
&(-1)^{n-1} d_n^2 \left[\prod_{s=1}^{n-1} \frac{1}{r_s} + \Delta^2 \sum_{k=1}^{n-1} \frac{1}{b_k^2}\left(\prod_{s=1}^{k} r_s\right)\left(\prod_{s=k+1}^{n-1} \frac{1}{r_s}\right)\right] + (-1)^{n-1} \Delta^2 \prod_{s=1}^{n-1} r_s\,.
\end{aligned} \tag{23}$$

Finally, if we replace the expression of the quantity $\overset{\circ}{\nu}_i$ (see formula (35) in [2]) as well as the expressions (22) and (23) into the equality (19), for the values of indeces $1 \le j \le n-1$ and $i = 1,2,\dots,j$, we get

$$z_{ij} = \frac{(-1)^{i+j}\left[\prod_{s=1}^{i-1} \frac{1}{r_s} + \Delta^2 \sum_{k=1}^{i-1} \frac{1}{b_k^2}\left(\prod_{s=1}^{k} r_s\right)\left(\prod_{s=k+1}^{i-1} \frac{1}{r_s}\right)\right] \cdot \frac{d_n^2}{d_j} \prod_{s=j}^{n-1} \frac{1}{r_s}}{d_n^2 \left[\prod_{s=1}^{n-1} \frac{1}{r_s} + \Delta^2 \sum_{k=1}^{n-1} \frac{1}{b_k^2}\left(\prod_{s=1}^{k} r_s\right)\left(\prod_{s=k+1}^{n-1} \frac{1}{r_s}\right)\right] + \Delta^2 \prod_{s=1}^{n-1} r_s}\,. \tag{24}$$

• ***Case*** $i = j+1, j+2, \dots, n$.

Using the expressions for the entries x_{ij} of the matrix $L(\varepsilon)^{-1}$ (formula (42) in [2]), from (15) we have

$$y_{ij}(\varepsilon) = t\mu_i(\nu_j d_j + \nu_{j+1} b_j). \tag{25}$$

Then, having the representations of the quantities ν_i (see (33) in [2]), we get

$$\nu_j d_j + \nu_{j+1} b_j = (\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j) + O(\varepsilon). \tag{26}$$

Substituting the expression (26) as well as the representations of the quantities μ_i and t (see (27) and (39) in [2]) into the right hand side of the equality (25), we obtain

$$y_{ij}(\varepsilon) = \frac{(\overset{\circ}{\mu}_i + O(\varepsilon))((\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j) + O(\varepsilon))}{\overset{\circ}{t} + O(\varepsilon)}.$$

Calculating the limit in the last equality as $\varepsilon \to +0$, according to (14), we find

$$z_{ij} = \frac{\overset{\circ}{\mu}_i (\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j)}{\overset{\circ}{t}}, \quad i = j+1, j+2, \dots, n, \tag{27}$$

or, by analogy with (19), $z_{ij} = \dfrac{\overset{\circ}{\mu}_i (\overset{\circ}{\nu}_j d_j + \overset{\circ}{\nu}_{j+1} b_j)}{d_n^2 \overset{\circ}{\nu}_n + \Delta^2 \alpha_1}, \quad i = j+1, j+2, \dots, n.$

From here, performing calculations similar to those that led to the formula (24), we obtain

$$z_{ij} = \frac{(-1)^{i+j+1}\left[\prod\limits_{s=i}^{n-1} r_s + d_n^2 \sum\limits_{k=i}^{n-1} \frac{1}{d_k^2}\left(\prod\limits_{s=i}^{k-1} r_s\right)\left(\prod\limits_{s=k}^{n-1} \frac{1}{r_s}\right)\right] \cdot \frac{\Delta^2}{d_j} \prod\limits_{s=1}^{j-1} r_s}{d_n^2\left[\prod\limits_{s=1}^{n-1} \frac{1}{r_s} + \Delta^2 \sum\limits_{k=1}^{n-1} \frac{1}{b_k^2}\left(\prod\limits_{s=1}^{k} r_s\right)\left(\prod\limits_{s=k+1}^{n-1} \frac{1}{r_s}\right)\right] + \Delta^2 \prod\limits_{s=1}^{n-1} r_s} \tag{28}$$

for the of indeces $1 \le j \le n-1$ and $i = j+1, j+2, \dots, n$.

So it remains to deduce expressions for the entries of the last column of the matrix Z. As follows from (13), the entries $y_{in}(\varepsilon)$ of the matrix $Y(\varepsilon)$ are calculated by the rule $y_{in}(\varepsilon) = x_{in} d_n,\ i = 1, 2, \dots, n$. According to the formula (43), from [2] we have $x_{in} = \mu_n \nu_i t$, and since $\mu_n = 1$ (see (8) in [1]) we get $y_{in}(\varepsilon) = \nu_i t d_n,\ i = 1, 2, \dots, n$. Thus, a similar argument that led us to the expressions (18) and (24), we find that

$$z_{in} = \frac{d_n \overset{\circ}{\nu}_i}{\overset{\circ}{t}}, \quad i = 1, 2, \dots, n, \tag{29}$$

or otherwise

$$z_{in} = \frac{(-1)^{n-i} d_n \left[\prod\limits_{s=1}^{i-1} \frac{1}{r_s} + \Delta^2 \sum\limits_{k=1}^{i-1} \frac{1}{b_k^2}\left(\prod\limits_{s=1}^{k} r_s\right)\left(\prod\limits_{s=k+1}^{i-1} \frac{1}{r_s}\right)\right]}{d_n^2\left[\prod\limits_{s=1}^{n-1} \frac{1}{r_s} + \Delta^2 \sum\limits_{k=1}^{n-1} \frac{1}{b_k^2}\left(\prod\limits_{s=1}^{k} r_s\right)\left(\prod\limits_{s=k+1}^{n-1} \frac{1}{r_s}\right)\right] + \Delta^2 \prod\limits_{s=1}^{n-1} r_s}, \tag{30}$$

where $i = 1, 2, \dots, n$.

Remark. It is easy to see that the formula (30) can be “incorporated” in the formula (24), if we extend the last one to the case $j = n$.

Summarizing the considerations of the section, namely having the expressions (12),(24) and (28), we get the following statement.

Lemma 1 **[Case A]**. For the matrix $Z = [z_{ij}]_{n\times n}$ defined in (10) we have:

if $n = 1$, then

$$Z = \left[\frac{d_1}{d_1^2 + \Delta^2}\right]_{1\times 1};$$

if $n \geq 2$, then:

a) for the indeces $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, j$ the entries z_{ij} are computed by the formula (24);

b) for the indeces $j = 1, 2, \dots, n-1$ and $i = j+1, j+2, \dots, n$ the entries z_{ij} are computed by the formula (28).

Now consider the matrix $H = [h_{ij}]_{n\times l}$. We introduce the matrix

$$L(\varepsilon)^{-1} B^T \equiv W(\varepsilon) = [w_{ij}(\varepsilon)]_{n\times n}. \tag{31}$$

By the definition (11) of the matrix H we have

$$h_{ij} = \lim_{\varepsilon\to+0} w_{ij}(\varepsilon), \quad i = 1, 2, \dots, n, \ j = 1, 2, \dots, l. \tag{32}$$

It can be easily seen that the entries of the first $l-1$ columns of the matrix $W(\varepsilon)$ are zero. This implies that the corresponding columns of the matrix H are also zeros. The entries of the last column of the matrix $W(\varepsilon)$ are written as follows:

$$w_{il}(\varepsilon) = x_{i1}\Delta, \quad i = 1, 2, \dots, n.$$

Since $x_{i1} = \mu_i \nu_1 t$ (see formula (42) in [2]) and $\nu_1 = 1$ (see (9) in [1]), then

$$w_{il}(\varepsilon) = \mu_i t\Delta, \quad i = 1, 2, \dots, n.$$

Thus, by the argument leading us to the expressions (18) and (24), we find that

$$h_{il} = \overset{\circ}{\mu}_i \Delta / \overset{\circ}{t}, \quad i = 1, 2, \dots, n, \tag{33}$$

or otherwise

$$h_{il} = \frac{(-1)^{i+1}\Delta\left[\prod\limits_{s=i}^{n-1} r_s + d_n^2 \sum\limits_{k=i}^{n-1}\frac{1}{d_k^2}\left(\prod\limits_{s=i}^{k-1} r_s\right)\left(\prod\limits_{s=k}^{n-1}\frac{1}{r_s}\right)\right]}{d_n^2\left[\prod\limits_{s=1}^{n-1}\frac{1}{r_s} + \Delta^2\sum\limits_{k=1}^{n-1}\frac{1}{b_k^2}\left(\prod\limits_{s=1}^{k} r_s\right)\left(\prod\limits_{s=k+1}^{n-1}\frac{1}{r_s}\right)\right] + \Delta^2\prod\limits_{s=1}^{n-1} r_s}, \tag{34}$$

where $i = 1, 2, \dots, n$.

Having (12) and (34), we get.

Lemma 2 **[Case A]**. The matrix $H = [h_{ij}]_{n\times l}$ defined in (11) satisfies the relationes:

if $n = 1$, then

$$H = \left[0\dots 0\,\frac{\Delta}{d_1^2+\Delta^2}\right]_{1\times l},$$

if $n \geq 2$, then:

a) $h_{ij} = 0$ for the indexes $j = 1,2,\dots,l-1$ and $i = 1,2,\dots,n$;

b) the entries h_{il}, $i = 1,2,\dots,n$, are computed by the formula (34).

Intermediate formulae and relations obtained in the section and in the second part of this work [2] allow us to propose the following numerical algorithm to compute the matrices Z and H.

Algorithm Z, H/CaseA $(A,\Delta,n,l \Rightarrow Z,H)$

If $n = 1$, then

$$Z = \left[\frac{d_1}{d_1^2+\Delta^2}\right]_{1\times 1}, \quad H = \left[0\dots 0\,\frac{\Delta}{d_1^2+\Delta^2}\right]_{1\times l}.$$

If $n \geq 2$, then

1. Compute the quantities r_s (see (20) from [2]):

$$r_s = b_s/d_s, \quad s = 1,2,\dots,n-1; \quad r_0 = r_1 = 1\,.$$

2. Compute the quantities α_i, $i = 1,2,\dots,n-1$ (see (32) from [2]):

$$\alpha_{n-1} = -r_{n-1}; \quad \alpha_i = -r_i\alpha_{i+1}, \quad i = n-2,n-3,\dots,1.$$

3. Compute the quantities β_i, $i = 1,2,\dots,n-1$ (see (38) from [2]):

$$\beta_1 = -r_1; \quad \beta_{i+1} = -r_{i+1}\beta_i, \quad i = 1,2,\dots,n-2.$$

4. Compute the quantities $\overset{\circ}{\mu}_i$, $i = 1,2,\dots,n$ (see (28) and (30) from [2]):

$$\overset{\circ}{\mu}_n = 1; \quad \overset{\circ}{\mu}_i = -r_i\,\overset{\circ}{\mu}_{i+1} + \frac{d_n^2}{d_i^2}\,\frac{1}{\alpha_i}, \quad i = n-1,n-2,\dots,1.$$

5. Compute the quantities $\overset{\circ}{\nu}_i$, $i = 1,2,\dots,n$ (see (34) and (36) from [2]):

$$\overset{\circ}{\nu}_1 = 1; \quad \overset{\circ}{\nu}_{i+1} = -\frac{1}{r_i}\,\overset{\circ}{\nu}_i + \frac{\Delta^2}{b_i^2}\,\beta_i, \quad i = 1,2,\dots,n-1.$$

6. Compute the quantity $\overset{\circ}{t}$ (see (40) from [2]):

$$\overset{\circ}{t} = d_n^2\,\overset{\circ}{\nu}_n + \Delta^2\alpha_1\,.$$

7. Compute the entries from the upper triangular part of the matrix Z (see (18)):

for the values $j = 1,2,\dots,n-1$:

$$z_{ij} = \overset{\circ}{\nu}_i\,(\overset{\circ}{\mu}_j\,d_j + \overset{\circ}{\mu}_{j+1}\,b_j)/\,\overset{\circ}{t}, \quad i = 1,2,\dots,j\,.$$

8. Compute the entries from the lower triangular part of the matrix Z (see (27)):

for the values $j = 1, 2, \ldots, n-1$:

$$z_{ij} = \overset{\circ}{\mu}_i \, (\overset{\circ}{\nu}_j \, d_j + \overset{\circ}{\nu}_{j+1} \, b_j) / \overset{\circ}{t}, \quad i = j+1, j+2, \ldots, n.$$

9. Compute the entries of the last column of the matrix Z (see (29)):

$$z_{in} = d_n \, \overset{\circ}{\nu}_i \, / \, \overset{\circ}{t}, \quad i = 1, 2, \ldots, n.$$

10. Compute the entries of the matrix H (see (33)):

$$h_{ij} = 0, \quad j = 1, 2, \ldots, l-1, \, i = 1, 2, \ldots, n;$$
$$h_{il} = \overset{\circ}{\mu}_i \, \Delta / \overset{\circ}{t}, \quad i = 1, 2, \ldots, n.$$

End

Direct calculations show that the numerical implementation of the algorithm **Z,H/CaseA** requires $n^2 + O(n)$ arithmetical operations. Thus the algorithm may be considered as an optimal one.

Case B. This case implies $n \geq 2$ and $d_1, d_2, \ldots, d_{n-1} \neq 0$, $d_n = 0$ (see [2]).

The difference between the cases A and B consists only in the value of the quantity d_n. In the first case we have $d_n \neq 0$, while in the second one $d_n = 0$. So we can use all the formulae and the expressions obtained in the previous section substituting $d_n = 0$.

As a consequence of the Lemma 1 we get the following statement.

L e m m a 3 [**Case B**]. For the matrix $Z = [z_{ij}]_{n \times n}$ defined in (10) we have:

a) $z_{ij} = 0$ for the indeces $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, j$;

b) for the indeces $j = 1, 2, \ldots, n-1$ and $i = j+1, j+2, \ldots, n$ the entries z_{ij} are computed by the formula

$$z_{ij} = \frac{(-1)^{i+j+1}}{d_j} \prod_{s=j}^{i-1} \frac{1}{r_s}. \tag{35}$$

The computing process of the lower triangular part entries of the matrix Z can be organized as follows. Consider fixed value of the index j from the range $1 \leq j \leq n-1$. For $i = j+1$, from (35) we get

$$z_{j+1\,j} = \frac{1}{d_j r_j} = \frac{1}{b_j}. \tag{36}$$

Further, for the subsequent values $i = j+2, \; j+3, \ldots, n$, again due to (35) the following relation holds:

$$z_{ij} = -\frac{z_{i-1\,j}}{r_{i-1}}. \tag{37}$$

Consider the matrix H. Setting $d_n = 0$, as a consequence of the Lemma 2 we obtain the following statement.

L e m m a 4 [**Case B**]. For the matrix $H = [h_{ij}]_{n \times l}$ defined in (11) we have:

a) $h_{ij} = 0$ for the indeces $j = 1, 2, \ldots, l-1$ and $i = 1, 2, \ldots, n$;

b) the entries h_{il} are computed by the formula

$$h_{il} = \frac{(-1)^{i+1}}{\Delta} \prod_{s=1}^{i-1} \frac{1}{r_s}, \quad i = 1, 2, \ldots, n. \tag{38}$$

The computation of the last column of the matrix H can also be carried out by a recurrence relation. If $i = 1$, then from (38) we have

$$h_{1l} = \frac{1}{\Delta}. \tag{39}$$

One can easily obtain from (38), that for the values $i = 2, 3, \ldots, n$ we have

$$h_{il} = -\frac{h_{i-1\,l}}{r_{i-1}}. \tag{40}$$

Having the formulae and relations obtained above, we can write the following algorithm to compute the entries of the matrices Z and H.

Algorithm Z,H/CaseB $(A, \Delta, n, l \Rightarrow Z, H)$

1. Compute the quantities r_s (see (20) from [2]):
   $$r_s = b_s/d_s, \quad s = 1, 2, \ldots, n-1; \quad r_0 = r_1 = 1.$$
2. Compute the entries of the upper triangular part of matrix Z (Lemma 3):
   for the values $j = 1, 2, \ldots, n$:
   $$z_{ij} = 0, \quad i = 1, 2, \ldots, j.$$
3. Compute the entries of the lower triangular part of matrix Z, see (36), (37):
   for the values $j = 1, 2, \ldots, n-1$:
   $$z_{j+1\,j} = 1/b_j;\ z_{ij} = -z_{i-1\,j}/r_{i-1},\ i = j+2, j+3, \ldots, n.$$
4. Compute the entries of the matrix H (see Lemma 4 and (39), (40)):
   $$h_{ij} = 0, \quad j = 1, 2, \ldots, l-1,\ i = 1, 2, \ldots, n;$$
   $$h_{1l} = 1/\Delta;\ h_{il} = -h_{i-1\,l}/r_{i-1},\ i = 2, 3, \ldots, n.$$

End

By direct calculations we find that the numerical implementation of the algorithm **Z,H/CaseB** requires $0.5n^2 + O(n)$ arithmetical operations. Thus, the algorithm may also be considered as an optimal one.

Case C. The case implies $n \geq 1$ and $d_1 = d_2 = \cdots = d_n = 0$ (see [2]).

If $n = 1$, the matrices Z and H have a very simple view:

$$Z = [0]_{1\times 1}, \quad H = \left[0 \ldots 0\, \frac{1}{\Delta}\right]_{1\times l}.$$

It obviously follows from (9), (10) and (11). The formulae for $n \geq 2$ are derived quite easily, using a technique, by which the cases A and B were examined. Therefore, let us formulate only the final results.

***Lemma* 5** [**Case C**]. For the matrix $Z = [z_{ij}]_{n\times n}$ defined in (10) we have:
if $n = 1$, then $Z = [0]_{1\times 1}$;
if $n \geq 2$, then $z_{i\,i-1} = \dfrac{1}{b_{i-1}}$, $i = 2, 3, \ldots, n$, and $z_{ij} = 0$ in the remaining cases.

***Lemma* 6** [**Case C**]. For the matrix $H = [h_{ij}]_{n\times l}$ defined in (11) we have: $h_{1l} = \dfrac{1}{\Delta}$ and $h_{ij} = 0$ in the remaining cases.

Due to the simplicity of the expressions for the entries of the matrices Z and H there is no need to write a special algorithm. We just point out that the computation of these matrices requires n arithmetical operations.

Concluding Remarks. Summarizing the preliminary results obtained in this paper as well as in previous papers [1, 2], in the next final part of the study we will give definitive solution to the Moore–Penrose inversions problem for singular upper bidiagonal matrices.

Received 12.01.2016

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